The Real-Quaternionic Indicator

and it's relation with the Frobenius-Schur indicator

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Outline

Definitions

Definition (Frobenius-Schur indicator)

(π, V) irrep of G, and $\pi \cong \pi^*$. Then $\exists B : V \times V \to \mathbb{C}$, which is G-inv and bilinear. Define:

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$$\delta(\pi) = sgn(c)$$

$\delta(\pi) = 1$ iff π is of real type;
$\delta(\pi) = -1$ iff π is of quaternionic type.

Existing Results

Theorem (Bourbaki)

$G(\mathbb{C})$ conn red cx Lie group, π finite-dim'l and $\pi \cong \pi^*$. Then

$$\varepsilon(\pi) = \chi_\pi(z_\rho)$$

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  Reduced the problem to fundamental representations.
- Jacques Tits 1967 "Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen"
  Listed values of $\delta(\lambda)$, where λ is a fundamental representations of simple Lie groups.

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Define a map $\mathcal{J} : V \to V$ such that:

$$B(v, w) = \langle v, \mathcal{J}(w) \rangle, \quad \forall v, w \in V$$

$\mathcal{J}$ is G-inv, conjugate linear and non-zero
$\Rightarrow \mathcal{J}^2(v) = cv, \forall v \in V$ for some $c \in \mathbb{R}^*$.

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Short calculation $\Rightarrow \varepsilon(\pi)\delta(\pi) = sgn\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\langle w, v \rangle}\right)$
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FS indicator

Consider bilinear form:

$$B(v,w) = v^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w$$

It is skew-symmetric $B(v,w) = -B(w,v)$ and G-invariant

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T$$

Therefore $\varepsilon(\pi) = -1$.

Outline

Main Theorem

Theorem (C.)

Let $(\pi, V) \cong (\overline{\pi}, \overline{V})$ *be finite-dim'l irrep a real reductive Lie group* G*, then*

1. π *Hermitian* $\Rightarrow \delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$
2. π *non-Hermitian* $\delta(\pi) = \varepsilon(\widetilde{\pi})\chi_\pi(x^2)$*, where* $\widetilde{\pi} = Ind_G^{{}^\gamma G}\pi$

x *is the "strong real form" of* G*, which means* $Ad(x) = \theta$*,* $x^2 \in Z(G)$*,* θ *is the Cartan involution of* G*.* ${}^\gamma G$ *is a extended group of* G*.*

$SL(2, \mathbb{R})$

Let $G = SL(2, \mathbb{R})$, the Cartan involution of $SL(2, \mathbb{R})$ is

$$\theta(g) = (g^T)^{-1}$$

The strong real form of $SL(2, \mathbb{R})$ is

$$x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Note $x^2 = -I$

$$1 = \delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2) = -1 \cdot -1$$

π is Hermitian

Proof of main theorem

Let $\langle , \rangle$ denote a G-invariant Hermitian form on V,
B denote a G-invariant bilinear form.
Define $\mathcal{J} : V \to V$ like before:

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Again we have

$$\varepsilon(\pi)\delta(\pi) = sgn\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\langle w, v \rangle}\right)$$

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Now what?

c-invariant Hermitian Form

Ordinary inv Hermitian form satisfies:

$$\langle \pi(g)v, w\rangle = \langle v, \pi(g^{-1})w\rangle$$

σ = real structure, a σ-invariant Hermitian form satisfies:

$$\langle \pi(g)v, w\rangle^{\sigma} = \langle v, \sigma(\pi(g^{-1}))w\rangle^{\sigma}$$

σ_0 given by the real form $G \rightsquigarrow \langle , \rangle$
σ_c given by the compact real form of $G(\mathbb{C}) \rightsquigarrow \langle , \rangle^c$.

Relation between $\langle , \rangle$ and $\langle , \rangle^c$

Proposition (Adams, van Leeuwen, Trapa, Vogan)

Let G be equal rank, π is finite-dim'l. Let $\langle , \rangle^c$ be a pos-def c-inv Hermiatian form. The form $\langle , \rangle$ defined as:

$$\langle v, w \rangle := \mu^{-1} \langle x \cdot v, w \rangle^c$$

is an ordinary Hermitian form.
x is the "strong real form", i.e., $Ad(x) = \theta$, Cartan involution, and $x^2 \in Z(G)$. μ is a square root of $\chi_\pi(x^2)$

G equal rank, π Hermitian

Rewrite $\varepsilon(\pi)\delta(\pi) = sgn\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle}{\langle w, v\rangle}\right)$ in terms of $\langle , \rangle^c$:

$$\varepsilon(\pi)\delta(\pi) = sgn\left(\mu^{-2}\frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w)\rangle^c}{\overline{\langle x \cdot v, w\rangle^c}}\right)$$

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G equal rk $\Rightarrow x \in G \Rightarrow x \cdot \mathcal{J}(v) = \mathcal{J}(x \cdot v)$.

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Set $w = x \cdot v \Rightarrow RHS = sgn(\mu^{-2}) \Rightarrow$

$$\delta(\pi) = \varepsilon(\pi)\mu^2 = \varepsilon(\pi)\chi_\pi(x^2)$$

G unequal rank, π Hermitian

θ is outer, the strong real form $x \in {}^{\gamma}G \backslash G$, where γ is distinguished involution in the inner class of θ.

Proposition

Suppose G is unequal rank, $\pi \cong \pi^* \cong \overline{\pi} \cong \pi^h$ then

1. *π extends irreducibly to $\pi_{\pm}$*
2. $\pi_{\pm} \cong (\pi_{\pm})^* \cong \overline{\pi_{\pm}} \cong (\pi_{\pm})^h$
3. *$\exists \langle , \rangle$ and $\langle , \rangle^c$ on $\pi_{\pm}$. They admit the same equation as before:*
$$\langle v, w \rangle = \mu^{-1} \langle x \cdot v, w \rangle^c$$

Use the same $\mathcal{J}$ and similar argument to show $\delta(\pi) = \varepsilon(\pi)\chi_{\pi}(x^2)$.

Outline

π Infinite-Dim'l

Similar arguments, using $\langle , \rangle^c$ pos-def on Lowest K-types.

Theorem

Let G be equal rank real reductive algebraic group, (π, V) is an infinite-dim'l $(\mathfrak{g}, K)$-module with real infinitesimal character. Suppose irrep $\pi \cong \pi^* \cong \overline{\pi} \cong \pi^h$, then

$$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$$

Outline

Signature of Hermitian Forms

Theorem (Adams-Cui)

Let G be an equal rank semisimple group, (π, V) be a finite-dim'l irrep of G, and π is Hermitian. Suppose $h = \langle , \rangle$ be a G-invariant Hermitian form on V, then

$$sgn(h) = \mu^{-1}\Theta_\pi(x)$$

where x is the strong real form of G, μ is a square root of $\chi_\pi(x^2)$, Θ_π is the global character of π.

Proof

Setting:

$\{v_1, \cdots, v_n\}$ basis of V consisting of T-weight vectors.
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Calculation:

We can define $\langle , \rangle^c$ such that $\{v_i\}$ is orthonormal w.r.t. $\langle , \rangle^c$.
Fact: we can assume $x \in T$, where $T = H^\theta$, H is a Cartan subgroup of G.

$\langle v_i, v_i \rangle = \mu^{-1} \langle x \cdot v_i, v_i \rangle^c = \mu^{-1} \lambda_i(x)$

$\Rightarrow \langle v_i, v_i \rangle^2 = \mu^{-2} \chi_\pi(x^2) = 1 \Rightarrow \langle v_i, v_i \rangle = \pm 1$

$sgn(h) = \sum_i \langle v_i, v_i \rangle = \sum_i \mu^{-1} \lambda_i(x) = \mu^{-1} \Theta_\pi(x)$

Thank you!