A BRIEF SURVEY OF THE FINITE HARMONIC OSCILLATOR AND ITS APPLICATIONS

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Abstract. A finite oscillator system was introduced by Gurevich, Hadani and Sochen. This is a survey of how the system is constructed using Weil representation on the group $\mathrm{SL}(2, \mathbb{F}_p)$ and its applications on discrete radar and CDMA system. Finally, explicit algorithms for computing the finite split and non-split oscillator systems $\mathfrak{S}^s$ and $\mathfrak{S}^{ns}$ are described.

1. Introduction

1.1. An Overview.

The Weil representation was constructed by André Weil [13]. It appears that this representation is a central object in mathematics, linking various topics in mathematics and physics together, including classical invariant theory, the theory of theta functions and automorphic forms, harmonic analysis, and quantum mechanics. In particular, the Weil representation of the group $\mathrm{Sp}(V, \omega)$ can be realized on the Hilbert space $\mathcal{H} = L^2(L, \mathbb{C})$, where L is a linear space and $V = L \times L^*$. In this realization, elements of the group Sp act by certain kinds of generalized Fourier transforms. In particular, there exists a specific element $w \in \mathrm{Sp}(V, \omega)$, called the *Weyl element*, whose action is given, up to normalization, by the standard Fourier transform. From this perspective, the classical theory of harmonic analysis seems to be devoted to the study of a particular operator in the Weil representation.

1.2. Motivation.

One-dimensional *digital signals* might be considered as complex-valued functions on the finite field $\mathbb{F}_p$, the parameter is denoted t and is referred to as *time*. In this section, we consider only digital signals, without ambiguity, we refer to them as signals. The space of signals $\mathcal{H} = \mathbb{C}(\mathbb{F}_p)$ is a Hilbert space with the Hermitian product given by

$$< \phi, \varphi >= \sum_{t \in \mathbb{F}_p} \phi(t)\overline{\varphi(t)}$$

A central problem is to construct interesting and useful systems of signals $\mathfrak{S}$. Here's a partial list of desired properties that $\mathfrak{S}$ wishes to satisfy:

• Signals are weakly correlated: for every $\phi \neq \varphi \in \mathfrak{S}$

$$| < \phi, \varphi > | \ll 1$$

During the transmission process, a signal φ may be distorted in various ways. Two basic types are *time shift* $\varphi(t) \mapsto \mathrm{L}_\tau\varphi(t) = \varphi(t+\tau)$ and *phase shift* $\varphi(t) \mapsto \mathrm{M}_\omega\varphi(t) = e^{\frac{2\pi i}{p}\omega t}\varphi(t)$, where $\tau, \omega \in \mathbb{F}_p$. A general distortion is of the type $\varphi \mapsto \mathrm{M}_\omega \mathrm{L}_\tau \varphi$, therefore a natural requirement would be:

- Stronger version of weak correlation: for every $\phi \neq \varphi \in \mathfrak{S}$

$$| < \phi, \mathrm{M}_\omega \mathrm{L}_\tau \varphi > | \ll 1$$

Due to technical restrictions, the system requires an additional condition:

- Low peak-to-average power ratio, i.e., for every $\varphi \in \mathfrak{S}$ with $\|\varphi\|_2 = 1$

$$\max\{|\varphi(t)| : t \in \mathbb{F}_p\} \ll 1$$

Several schemes for digital communication require that:

- The above properties are preserved under Fourier transform.

1.3. **Structure of the Paper.**

In section 2, several basic notions from representation theory are introduces. In section 3, we take the representation theoretic point of view to construct the well-known Heisenberg system. Section 4 will dedicate to constructing the novel system, called the oscillator system, using the Weil representation. In the last section, applications on discrete radar and CDMS system will be explained.

2. Preliminaries for Representation Theory

In this section, several fundamental notions from representation theory will be explained. The field we will be working on in this section is $\mathbb{F}_q$, a finite field of q elements, where q is odd.

2.1. **The Heisenberg representation.**

Let (V, ω) be a $2N$-dimensional symplectic vector space over the finite field $\mathbb{F}_q$. ω is a non-degenerate skew symmetric bilinear form on V. The *Heisenberg group* $H = H(V, \omega)$ is the set $V \times \mathbb{F}_q$ equipped with the following multiplication rule:

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\omega(v, v')). \tag{1}$$

Considering V as an additive group, H admits a nontrivial central extension

$$0 \to \mathbb{F}_q \to H \to V \to 0$$

The center of H is denoted as $Z = Z(H) = \{(0, z) : z \in \mathbb{F}_q\}$. Fix a non-trivial central character $\psi : Z \to \mathbb{C}^\times$, we have the following fundamental theorem:

Theorem 1. *(Stone-von Neumann). There exists a unique (up to isomorphism) irreducible unitary representation* $(\pi, H, \mathcal{H})$ *with the center acting by* ψ*, i.e.,* $\pi(z) = \psi(z)Id_{\mathcal{H}}$ *for every* $z \in Z$*.*

We call this representation the *Heisenberg Representation* associated with the central character ψ.

2.2. The Weil Representation.

Let $G = \mathrm{Sp}(V, \omega)$ be the group of symplectic linear automorphisms of V. The group G acts acts by group automorphism on the Heisenberg group through its tautological action on the vector space V, i.e.,

$$(v, z) \mapsto (g(v), z), \quad (v, z) \in H, g \in G$$

This induces a right action of G on the category $\mathrm{Rep}(H)$ of representation of H:

$$\pi \mapsto \pi^g \text{ where } \pi^g(h) = \pi(g \cdot h)$$

It's clear that this action doesn't affect the central character, and it sends an irreducible representation to an irreducible one. This means π^g still satisfies the requirements of Stone-von Neumann theorem. Therefore

Theorem 2. *For every element $g \in G$, where π is the Heisenberg representation we have*

$$\pi^g \cong \pi$$

Define $\widetilde{\rho}(g) : \mathcal{H} \to \mathcal{H}$ an intertwiner which realizes this isomorphism. i.e.,

$$\widetilde{\rho}(g) \circ \pi(h) = \pi^g(h) \circ \widetilde{\rho}(g) \tag{2}$$

From Shur's lemma, $\widetilde{\rho}(g)$ is uniquely determined by g up to a scalar. In fact, $\{\widetilde{\rho}(g) : g \in G\}$ forms a projective representation $\widetilde{\rho} : G \to \mathrm{PGL}(\mathcal{H})$.

By the following theorem, $\widetilde{\rho}$ can be linearized into a representation [11] which is denoted $\rho : G \to \mathrm{GL}(\mathcal{H})$. This is called the *Weil representation.*

Theorem 3. *There exists a canonical representation*[5]

$$\rho : G \to GL(\mathcal{H})$$

satisfying the euqation (2)

Remark 1. *A. Weil showed in* [13] *that the projective representation $\widetilde{\rho}$ cannot be lifted to an ordinary representation $G \to GL(\mathcal{H})$ for a local field $F \neq \mathbb{C}$. In this case, $\widetilde{\rho}$ can be realized on some central extension of $Sp(V, \omega)$ which is proven to be the double cover:*

$$1 \to \mathbb{Z}_2 \to \widetilde{Sp} \to Sp \to 1$$

called the metaplectic cover.

3. The Heisenberg System

The Heisenberg system is a well-known system in signal processing, I will give a brief explanation about this system, taking the representation theoretic point of view.

3.1. **Construction.**

Let $\psi(t) = e^{\frac{2\pi i}{p}t}$, $V = \mathbb{F}_p \times \mathbb{F}_p$. $H = H(V)$ is the Heisenberg group on V. Define the *Heisenberg representation* $\pi : H \to U(\mathcal{H})$ as

$$\pi(\tau, \omega, z) = \psi(-\frac{1}{2}\tau\omega + z)M_\omega \circ L_\tau$$

Note: $\pi(\tau, 0, 0) = L_\tau$ and $\pi(0, \omega, 0) = M_\omega$.

To construct the Heisenberg system, consider maximal commutative subgroups in H. To every line $L \subset V$, that pass through the origin, one can associate a maximal commutative subgroup $A_L\{(l, 0) \in V \times \mathbb{F}_p : l \in L\}$. Without ambiguity, we identify A_L with L.

Restrict π to L gives a decomposition of the Hilbert space $\mathcal{H}$ into a direct sum of one-dimensional subspaces

$$\mathcal{H} = \oplus_\chi \mathcal{H}_\chi$$

where χ runs in the set $L^\vee$ of complex-valued characters of L. The subspace $\mathcal{H}_\chi$ consists of vectors $\phi \in \mathcal{H}$ such that $\pi(l)\varphi = \chi(l)\varphi$. In other words, the space $\mathcal{H}_\chi$ consists of common eigenvectors with respect to the commutative system of operators $\{\pi(l)\}_{l\in L}$ such that the operator $\pi(l)$ has eigenvalue $\chi(l)$.

Choosing a unit vector $\phi_\chi \in \mathcal{H}_\chi$ for every $\chi \in L^\vee$ we obtain an orthonormal basis $\mathcal{B}_L = \{\phi_\chi : \chi \in L^\vee\}$. The system of signals consisting of a union of orthonormal bases

$$\mathfrak{S}_H = \{\phi \in \mathcal{B}_L : L \subset V\}$$

A brief calculation shows that the system $\mathfrak{S}_H$ consists of $p(p + 1)$ signals.

3.2. **Properties of the Heisenberg System.**

It's convenient to introduce some general notions which we will be using throughout this paper.

Given two signals $\phi, \varphi \in \mathcal{H}$, their *matrix coefficient* is the function $m_{\phi,\varphi} : H \to \mathbb{C}$ given by

$$m_{\phi,\varphi}(h) =< \phi, \pi(h)\varphi > .$$

In coordinates, if we write $h = (\tau, \omega, z)$ then $m_{\phi,\varphi}(h) = \psi(-\frac{1}{2}\tau\omega + z) < \phi, M_\omega \circ L_\tau\varphi >$. When $\phi = \varphi$, the function

$$A_\varphi = m_{\varphi,\varphi}$$

is called the *ambiguity function* of the vector φ.

$\mathfrak{S}_H$ satisfies the following properties[10][8]:

1) *Autocorrelation.* For every signal $\varphi \in \mathcal{B}_L$ the function $|A_\varphi|$ is the characteristic function of the line L, i.e.,

$$|A_\varphi| = \begin{cases} 0, & v \notin L \\ 1, & v \in L \end{cases}$$

2) *Cross-Correlation.* For every $\phi \in \mathcal{B}_L$ and $\varphi \in \mathcal{B}_M$, where $L \neq M$, we have

$$|m_{\varphi,\phi}(v)| \leq \frac{1}{\sqrt{p}}$$

3) *Supremum.* A signal $\varphi \in \mathfrak{S}_H$ is unimodular function, i.e., $|\varphi(t)| = \frac{1}{\sqrt{p}}$ for every $t \in \mathbb{F}_p$, in particular we have

$$\max\{|\varphi(t)| : t \in \mathbb{F}_p\} = \frac{1}{\sqrt{p}} \ll 1$$

4. The Oscillator System

4.1. the Weil representation on $\mathbf{SL}_2(\mathbb{F}_p)$.

$\mathcal{H}$ is still defined as $\mathbb{C}(\mathbb{F}_p)$. Let $Sp = \mathrm{SL}(2,\mathbb{F}_p) \cong \mathrm{Sp}(2,\mathbb{F}_p)$ which is generated by $s_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $n_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ where $a \in \mathbb{F}_p^{\times}$, $b \in \mathbb{F}_p$. In detail: for every element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp$

$$g = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ bd & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ ab^{-1} & 1 \end{pmatrix}, \text{ when } b \neq 0$$

$$g = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ ac & 1 \end{pmatrix}, \text{ when } b = 0$$

The Weil representation $\rho : Sp \to U(\mathcal{H})$ can be determined by $\rho(s_a)$, $\rho(n_b)$ and $\rho(\omega)$ which are defined in [7] as follows:

$$\rho(s_a)\varphi(t) = \sigma(a)\varphi(a^{-1}t) \tag{3}$$

$$\rho(n_b)\varphi(t) = \psi(-\frac{bt^2}{2})\varphi(t) \tag{4}$$

$$\rho(\omega)\varphi(l) = \frac{\nu}{\sqrt{p}} \sum_{s \in \mathbb{F}_p} \psi(ls)\varphi(s) \tag{5}$$

where σ is the *Legendre character*, $\sigma(a) = \left(\frac{a}{p}\right)$ and ν is a normalization constant. $\psi(x) = e^{\frac{2\pi i}{p}x}$. Note that the operator $\rho(\omega)$ is the Fourier transform.

For convenience, denote the operators $S_a = \rho(s_a)$, $N_b = \rho(n_b)$, and $F = \rho(\omega)$.

By the decomposition described above, the Weil representation of g is given by

$$\rho(g) = S_b \circ N_{bd} \circ F \circ N_{ab^{-1}}, \text{ when } b \neq 0 \tag{6}$$

$$\rho(g) = S_a \circ N_{ac}, \text{ when } b = 0 \tag{7}$$

4.2. The Maximal Tori in Sp.

To summarize the construction of this system, if we take the maximal tori $T \subset Sp$, the Hilbert space $\mathcal{H}$ will decompose into one dimensional sub-representations when restrict the Weil representation on T. If we take certain orthonormal vectors in the sub-representations for all maximal tori, the union will be the oscillator system $\mathfrak{S}$.

A maximal *algebraic torus* in Sp is a maximal commutative subgroup which becomes diagonalizable over the original field $\mathbb{F}_p$ or over the quadratic extension of $\mathbb{F}_p$. Any torus which is conjugated to the standard diagonal torus

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^{\times} \right\}$$

is called *split* torus. Any torus which is not conjugated to A in $\mathbb{F}_p$ but in $\mathbb{F}_{p^2}$ is called *non-split* torus. In fact, a split torus is a cyclic subgroup of Sp with order $p-1$, while a non-split torus is a cyclic subgroup of Sp with order $p+1$.

Since every maximal torus $T \subset Sp$ is a cyclic group and T acts semisimply on $\mathcal{H}$, decomposing it into a direct sum of character spaces $\mathcal{H} = \oplus \mathcal{H}_\chi$ over the characters (one dimensional sub-representations) of T. The multiplicities $m_\chi = \dim \mathcal{H}_\chi$ (cf.: [1][3][12]) will be described below.

4.3. The Character of the Weil Representation and the Multiplicities Formula.

The absolute value of the character $ch_\rho : Sp \to \mathbb{C}$ of the Weil representation was described in [9] , and the explicit formula was given in [4]:

$$ch_\rho(g) = \sigma((-1)^N \cdot \det(g - I)) \tag{8}$$

when $g \neq I$ and $N = 1$ in our case.

For a maximal torus $T \subset Sp$, denote $\sigma_T : T \to \mathbb{C}^\times$ the unique quadratic character of T.

Theorem 4. *(Multiplicities formula) We have $m_\chi = 1$ for any character $\chi \neq \sigma_T$. Moreover, $m_{\sigma_T} = 2$ or 0, depending on whether the torus T is split or non-split, respectively.*

Proof is given in [6]

The decomposition mentioned in section 3.3 is in fact

$$\mathcal{H} = \mathcal{H}_{\chi_1} \oplus \cdots \oplus \mathcal{H}_{\chi_{p-2}} \oplus \mathcal{H}_\sigma^2 \text{ (abuse of notation?) }, \quad \chi_i, \sigma : \mathbb{F}_{p-1} \to \mathbb{C}$$

for split tori, and

$$\mathcal{H} = \mathcal{H}_{\chi_1} \oplus \cdots \oplus \mathcal{H}_{\chi_p}, \quad \chi_i : \mathbb{F}_{p+1} \to \mathbb{C}, \chi_i \neq \sigma$$

for non-split tori

4.4. The Oscillator System.

For a given torus T, choosing a unit vector $\varphi_\chi \in \mathcal{H}_\chi$ for every χ we obtain a collection of orthonormal vector $\mathcal{B}_T = \{\varphi_\chi : \chi \in T^\vee, \chi \neq \sigma\}$. Note that when T is non-split, the system $\mathcal{B}_T$ is an orthonormal basis. Considering the union of all these collections, we obtain the oscillator system

$$\mathfrak{S}_O = \{\varphi \in \mathcal{B}_T : T \subset Sq \text{ is maximal torus }\}$$

It will convenient to separate the system $\mathfrak{S}_O$ into two subsystems, $\mathfrak{S}_O^s$ and $\mathfrak{S}_O^{ns}$, which correspond to the split tori and the non-split tori, respectively.

Since all split (non-split) tori are conjugated to each other, the number of split tori is $\#(Sp/N_s) = \frac{q(q+1)}{2}$; the number of non-split tori is $\#(Sp/N_{ns}) = q(q-1)$, where N_s (N_{ns}) is the normalizer group of some split (non-split) torus.[7]

The subsystem $\mathfrak{S}_O^s$ consists of $\frac{q(q+1)}{2}$ collections, each consisting of $q-2$ orthonormal vectors, altogether $\#\mathfrak{S}_O^s = \frac{q(q+1)(q-2)}{2}$. The non-split subsystem $\mathfrak{S}_O^{ns}$ consists of $q(q-1)$ collections each consisting of q orthonormal vectors, altogether $\#\mathfrak{S}_O^{ns} = q^2(q-1)$. The number of signals in the oscillator system is $\#\mathfrak{S}_O = \frac{3q^2-3q^2-2q}{2}$

The properties of $\mathfrak{S}_O$ are summarized bellow [7]:

• (Autocorrelations): For every $\varphi \in \mathcal{B}_T$

$$|A_\varphi(h)| = \begin{cases} 1, & h \in Z \\ \leq \frac{2}{\sqrt{q}}, & h \neq Z \end{cases}$$

• (Cross-Correlations): For every $\varphi \in \mathcal{B}_T$ and $\varphi' \in \mathcal{B}_{T'}$

$$|m_{\varphi,\varphi'}(h)| \leq \frac{4}{\sqrt{q}}$$

if $T = T'$, $\varphi \neq \varphi'$, there exists an improved estimate:

$$|m_{\varphi,\varphi'}(h)| \leq \frac{2}{\sqrt{q}}$$

• (Supermum): Let $S = (L, M)$ be the *Lagrangian splitting* of V, then for every $\varphi \in \mathcal{B}_T$

$$\sup_{x \in L} |\varphi(x)| \leq \frac{2}{\sqrt{q}}$$

• (Fourier Invariance) For all $\varphi \in \mathfrak{S}_O$, its Fourier transform $\hat{\varphi}$ is (up to multiplication by a unitary scalar) also in $\mathfrak{S}_O$.

Note: In fact, $\mathfrak{S}_O$ is closed under all operators in the Weil representation.

4.5. **Algorithm for the Oscillator System.**

4.5.1. *The split oscillator system* $\mathfrak{S}_O^s$. :

Consider the standard diagonal torus A in section 3.3, let $\mathcal{T} = \{gAg^{-1}; g \in Sp\}$. A direct calculation shows that every torus in $\mathcal{T}$ can be written as gAg^{-1} with

$$g = \begin{pmatrix} 1+bc & b \\ c & 1 \end{pmatrix}, b, c \in \mathbb{F}_p$$

Let's choose a set of elements of this form representing each torus in $\mathcal{T}$ exactly once and denote this set of representative elements by R.

A pseudocode is given as below [7]:

1) Choose a prime p.
2) Compute generator g_A for the standard torus A.
3) Diagonalize $\rho(g_A)$ and obtain the basis of eigenfunctions $\mathcal{B}_A$.
4) For every $g \in R$, compute the operator $\rho(g)$ in term of the Bruhat decomposition
5) Compute the vectors $\rho(g)\varphi$, for every $\varphi \in \mathcal{B}_A$ and obtain the basis $\mathcal{B}_{gAg^{-1}}$.

Note: The time complexity of the algorithm is $O(p^4 \log p)$.

4.5.2. *The non-split oscillator system* $\mathfrak{S}_O^{ns}$. The algorithm for $\mathfrak{S}_O^{ns}$ is given below: [2].

1) Choose a prime p.

2) Choose a non-square element $D \in \mathbb{F}_p$, and s and t such that $s + t\sqrt{D}$ is a primitive element of $\mathbb{F}_{p^2}$.

3) Let $g_D = \frac{1}{s^2 - Dt^2} \begin{pmatrix} s^2 + Dt^2 & -2st \\ -2stD & s^2 + Dt^2 \end{pmatrix}$. Diagonalize $\rho(g_D)$ to obtain $\mathcal{B}_T$

4) If $p \equiv 3 \pmod 4$, then

$$\mathfrak{S}_O^{ns} = \{S_a \circ N_{ac}(\varphi) : \varphi \in \mathcal{B}_T, 1 \leq a \leq \frac{p-1}{2}, 0 \leq c \leq p-1\}.$$

If $p \equiv 1 \pmod 4$, then

$$\mathfrak{S}_O^{ns} = \{S_a \circ N_{ac}(\varphi) : \varphi \in \mathcal{B}_T \cup F \circ \mathcal{B}_T, c \in \mathbb{F}_p^*, 0 \leq c \leq p-1\}$$

5. Applications

5.1. **Discrete Radar.**

The theory of discrete radar is closely related to the finite Heisenberg group H, as introduced in section 3. A radar sends a signal $\varphi(t)$ and obtains an echo $e(t)$, related by the transformation

$$e(t) = M_\omega L_\tau \varphi(t) = \pi(h_0)\varphi(t)$$

The goal is to reconstruct the target range and velocity which are encoded in $h_0 = (\tau, \omega, z)$ by $\tau = \frac{2X}{c}, \omega = -\frac{2\omega_0 v}{c}$ where X is the distance, v is the velocity.

It's easy to see that $|m_{\varphi,e}(h)| = |A_\varphi(h \cdot h_0)|$ and it obtains its maximum at h_o^{-1}. This suggests that a desired signal φ for discrete radar should admit an ambiguity function A_φ which is highly concentrated around $0 \in H$, which is the property satisfied by signals in $\mathfrak{S}_O$ (autocorrelation).

5.2. **Code-Division Multiple Access (CDMA).**

Let I be a collection of users, each holding a bit of information $b_i \in \mathbb{C}$, each user have a private signal $\varphi_i \in \mathcal{H}$ and then a message $u_i = b_i\varphi_i$. The transmission is carried through a single channel, therefore, the message received by the antenna is the sun

$$u = \sum_i u_i$$

The goal is to extract the individual bits b_i from the message u. b_i can be estimated by calculating the inner product

$$< \varphi_i, u > = \sum_i < \varphi_i, u_i > = b_i + \sum_{j \neq i} < \varphi_i, \varphi_j >$$

The last term is an additional noise. This is the standard scenario also called the *asynchronous scenario*. More complicated scenario appear in practice, e.g., *asynchronous scenario* and *phase-shift scenario*. In these scenarios, each message u_i is allowed to acquire an arbitrary distortion of the form

$$u_i(t) \mapsto e^{\frac{2\pi i}{p}\omega_i t} u_i(t + \tau_i)$$

Now the message is of the form

$$u_i = b_i \pi(h_i)\varphi_i$$

To extract b_i, compute the matrix coefficient

$$m_{\varphi_i,u}(h) = b_i A_{\varphi_i}(hh_i) + \#(J - \{i\})o(1)$$

where $J \subseteq I$

If the cardinality of J is not too big, then by evaluating $m_{\varphi_i,u}$ at $h = h_i^{-1}$, we can reconstruct the bit b_i.

The oscillator system can support order of p^3 users and is a stable system which is defined below:

Stability Conditions: Two unit signals $\phi \neq \varphi$ are called *stably cross-correlated* if $|m_{\phi,\varphi}(v)| \ll 1$ for every $v \in V$. A unit signal φ is called *stably autocorrelated* if $|A_\varphi(v)| \ll 1$, for every $v \neq 0$. A system $\mathfrak{S}$ of signals is called a *stable system* if every signal $\varphi \in \mathfrak{S}$ is stably autocorrelated and any two different signals $\phi, \varphi \in \mathfrak{S}$ are stably cross-correlated.

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