The Real-Quaternionic Indicator

Ran Cui

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Abstract

If an irreducible complex representation is self-conjugate, then it's either "real" or "quaternionic" (coming from a real representation or a quaternionic one). The Real-Quaternionic indicator is defined to be ± 1 accordingly. This indicator has application to mathematical physics, and many mathematicians have studied it. For instance, Iwahori [5] and Jacuqes Tits [8] studied the indicator for finite-dimensional representations of Lie algebras. In this paper, we study the indicator for infinite-dimensional representation of real reductive algebraic Lie groups. Our main result is a relation between the Real-Quaternionic indicator and the more familiar Frobenius-Schur indicator, which tells if the bilinear forms on a self-dual representation is symmetric or skew-symmetric. The main tool is the notion of c-invariant Hermitian form, originally developed in [2] in the study of unitary representations. In particular, we give a closed formula for the Real-Quaternionic indicator of a finite-dimensional self-conjugate representation.

1 Introduction

The Real-Quaternionic indicator (Definition 2.5 and Definition 2.6) was introduced by Iwahori in 1959 [5] as a tool for finding all real irreducible representations of a real Lie algebra. Jacuqes Tits gave a formula for the Real-Quaternionic indicator for finite-dimensional representation [8], also see [6]. However, these formulas are all case-by-case.

It is well known that the Real-Quaternionic indicator and the Frobenius-Schur indicator agree for finite groups, and this generalizes as usual to compact groups [4]. In this paper, we will give a new proof of this fact (Theorem 2.2). The new proof clearly suggests the reason behind this equality is the existence of a positive-definite invariant Hermitian form. This discovery urges us to focus on invariant Hermitian forms. For non-unitary representations, the invariant Hermitian forms are not positive-definite. So we turn our attention to the c-invariant Hermitian form (Definition 3.10) which approximates positive-definite Hermitian forms. From the relation between the ordinary invariant Hermitian form and the c-invariant Hermitian form, we deduce a relation between the two indicators in general. The main results of this paper are Theorem 3.11, Theorem 3.15, and Theorem 3.22. For the convenience of interested reader, we single out the finite-dimensional case in Section 4, and using the existing result for the Frobenius-Schur indicator to give a closed formula for the Real-Quaternionic indicator (Theorem 4.3).

2 Compact Lie Groups

In this self-contained section, we present the basic case of representations of compact groups. This serves as motivation for the general case.

To lay a solid foundation for the proofs in this paper, we will first give the definitions of some basic notions.

2.1 Definitions

In this section G will denote a compact group, and (π, V) a finite-dimensional irreducible representation of G. Basic representation theory tells us that π is finite-dimensional and unitary. We are interested in some operations on π, particularly, the dual, the conjugate and the Hermitian dual of π.

Definition 2.1 (Dual Representation)**.** The *dual representation* of π is the representation π^* on the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{C})$. The group G acts on V^* by

$$\pi^*(g)(f) = f \circ \pi(g^{-1}) \qquad \forall g \in G, f \in V^*.$$

A representation is said to be self-dual if $\pi \cong \pi^*$.

Remark 1. By this definition, it is easy to see that the double dual of a representation is canonically isomorphic to itself.

$$\pi^{**} \cong \pi.$$

Definition 2.2 (Conjugate Representation)**.** The *conjugate representation* of π is the representation $\overline{\pi}$ on the conjugate vector space $\overline{V} = \mathbb{C} \otimes_{\mathbb{C},\tau} V$ where $\tau : \mathbb{C} \to \mathbb{C}$ is a complex conjugation. The group G acts on $\overline{V}$ by

$$\overline{\pi}(g)(z \otimes_{\mathbb{C},\tau} v) = z \otimes_{\mathbb{C},\tau} \pi(g)v \qquad \forall g \in G, z \in \mathbb{C}, v \in V.$$

A representation is said to be self-conjugate if $\pi \cong \overline{\pi}$.

Definition 2.3 (Hermitian Dual Representation)**.** The *Hermitian dual* of π is the representation π^h on the Hermitian dual vector space

$$V^h = \{\xi : V \to \mathbb{C} | \xi(zv + yw) = \overline{z}\xi(v) + \overline{y}\xi(w), \forall z, y \in \mathbb{C}; v, w \in V\}.$$

The group G acts on V^h by

$$\pi^h(g)(\xi) = \xi \circ \pi(g^{-1}), \qquad \forall g \in G, \xi \in V^h.$$

A representation is said to be Hermitian if $\pi \cong \pi^h$.

Remark 2. The representation (π, V) being Hermitian is equivalent to the existence of a G invariant Hermitian form on V.

These three operations on a representation have the following relation:

Theorem 2.1. *Any two of the three operations, dual, conjugate, and Hermitian dual compose into the third.*

$$(\overline{\pi}, \overline{V}) = ([\pi^h]^*, [V^h]^*) = ([\pi^*]^h, [V^*]^h)$$

$$(\pi^*, V^*) = (\overline{[\pi^h]}, \overline{[\pi^h]}) = ([\overline{\pi}]^h, [\overline{V}]^h)$$

$$(\pi^h, V^h) = (\overline{[\pi^*]}, \overline{[\pi^*]}) = ([\overline{\pi}]^*, [\overline{V}]^*)$$

Remark 3. The above theorem foreshadows the relation among the Real Quaternionic indicator (which is defined for self-conjugate representations), the Frobenius Schur indicator (which is defined for self-dual representations), and the invariant Hermitian form of π.

Definition 2.4 (Frobenius-Schur Indicator)**.** Suppose (π, V) is an irreducible self-dual representation of G. Then there exists bilinear form B on V that is invariant under the action of G. The *Frobenius-Schur indicator* (also called the ε-indicator) is defined to be:

$$\varepsilon(\pi) = \begin{cases} 1 & B \text{ is symmetric} \\ -1 & B \text{ is skew-symmetric} \end{cases}.$$

Remark 4. The self-duality of π implies there exists invariant bilinear forms on V. One of these bilinear forms can be easily defined using the isomorphism between π and π^*, and any other invariant bilinear form on V is a complex multiple of it. Using Schur's Lemma, it is straightforward to prove that any such form must be either symmetric or skew-symmetric. For a fixed representation π, the invariant bilinear form is unique up to a complex scalar. Therefore the ε-indicator is well defined.

Definition 2.5 (Real-Quaternionic Indicator)**.** Suppose (π, V) is an irreducible self-conjugate representation of G. Then there exists a non-zero G invariant conjugate-linear map $\mathcal{J} : V \to V$. For such a map, there exists $c \in \mathbb{R}^*$ such that $\mathcal{J}^2(v) = cv$ for all $v \in V$. The *Real-Quaternionic indicator* (also called the δ-indicator) is defined to be:

$$\delta(\pi) = \operatorname{sgn}(c).$$

Remark 5. The isomorphism between π and $\overline{\pi}$ canonically defines a non-zero G invariant conjugate-linear map $\mathcal{J}$. Any other map that satisfies the same conditions will be a complex multiple of $\mathcal{J}$. Since $\mathcal{J}$ is conjugate linear, the sign of $(z\mathcal{J})^2$ for $z \in \mathbb{C}$ is the same as the sign of $\mathcal{J}^2$. Therefore the δ-indicator is well defined.

We give an equivalent definition to illustrate the origin of the name "Real-Quaternionic indicator".

Definition 2.6. Let (π, V) be an irreducible self-conjugate representation of G. We say (π, V) is of *real type* if there exists an irreducible real representation (π_0, W) of G with W a real vector space such that $\pi \cong \pi_0 \otimes \mathbb{C}$; we say (π, V) is of *quaternionic type* if there exists an irreducible quaternionic representation (ρ, U) with U an $\mathbb{H}$ vector space such that $\pi \cong \operatorname{Res}_{\mathbb{C}}^{\mathbb{H}} \rho$ (here Res denotes the restriction of scalars). The *Real-Quaternionic indicator* δ is defined to be:

$$\delta(\pi) = \begin{cases} 1 & \pi \text{ is of real type} \\ -1 & \pi \text{ is of quaternionic type} \end{cases}.$$

The main theorem of this section is the following.

Theorem 2.2. *Let G be a compact Lie group and (π, V) an irreducible self-conjugate representation of G. Then $\delta(\pi) = \varepsilon(\pi)$.*

Proof. Recall that any irreducible representation of a compact group is unitary, so there exists a non-degenerate G invariant positive-definite Hermitian form $\langle,\rangle$ on V. Moreover, π is Hermitian, i.e., $\pi \cong \pi^h$. Together with the assumption $\pi \cong \overline{\pi}$ and Theorem 2.1, we conclude $\pi \cong \pi^*$, i.e., π is self-dual. The ε-indicator on π is therefore defined.

In order to prove the equality in question, we first define a map $\mathcal{J} : V \to V$. The self-duality of π ensures the existence of a non-degenerate, G invariant bilinear form B on V. Define $\mathcal{J}$ by the condition:

$$B(v, w) = \langle v, \mathcal{J}(w)\rangle, \qquad \forall v, w \in V.$$

It is easy to verify that $\mathcal{J}$ is conjugate-linear, G-equivariant and non-zero. By definition, there exists $c \in \mathbb{R}^*$ such that $\mathcal{J}^2(v) = cv$ for all $v \in V$.

The following computation is the prototype for many proofs in this paper.

$$\begin{aligned}\langle \mathcal{J}(v), \mathcal{J}(w)\rangle = B(\mathcal{J}(v), w) &= \varepsilon(\pi)B(w, \mathcal{J}(v)) = \varepsilon(\pi)\langle w, \mathcal{J}^2(v)\rangle \\ &= \varepsilon(\pi)\langle w, cv\rangle = c \cdot \varepsilon(\pi)\langle w, v\rangle \qquad \forall v, w \in V\end{aligned}$$

This implies:

$$\varepsilon(\pi)\delta(\pi) = \varepsilon(\pi)\mathrm{sgn}(c) = \mathrm{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle}{\langle w, v\rangle}\right) \qquad \forall v, w \in V \text{ where } \langle v, w\rangle \neq 0.$$

Set $v = w$, the right hand side is 1 because $\langle,\rangle$ is positive-definite. Therefore $\varepsilon(\pi) = \delta(\pi)$. □

Remark 6. The reader might have noticed that we only used the compactness of the group to deduce the unitarity of the representation π. This proof can be done whenever the representation is unitary and the two indicators are well defined. In particular, if G is a real reductive Lie group and π is an irreducible self-conjugate representation of G, then $\varepsilon(\pi) = \delta(\pi)$. The contra-positive statement gives an interesting non-unitarity condition.

3 Real Reductive Algebraic Groups

In this section, G will denote a real reductive algebraic group.

Definition 3.1 (Real Reductive Algebraic Group)**.** A *real reductive algebraic group* G (which we will call "real group" for short) is the group of real points of a complex connected reductive algebraic group. In other words, given a real form σ of a complex group $G(\mathbb{C})$, $G = G(\mathbb{R}, \sigma) = G(\mathbb{C})^\sigma$.

Our goal is to establish a relation between the ε-indicator and the δ-indicator as we did in the compact case. Historically, the Frobenius-Schur indicator, i.e., the ε-indicator, is better understood than the δ-indicator. It has been defined in a larger setting also, for example, it is defined and studied for p-adic groups [7]. Building this bridge between the two gives a better understanding of the δ-indicator. For instance, we are able to give a closed formula for the δ-indicator of finite-dimensional self-conjugate representations of G using this relation and an existing formula for the ε-indicator.

It is also worth pointing out that because the result for compact groups is very well known, people often confuse the Real-Quaternionic indicator with the

Frobenius-Schur indicator or think that they are the same. The generic names researchers have been giving the Real-Quaternionic indicator didn't help with the misunderstanding either.

3.1 Motivational Examples

As a first example, We want to illustrate to the reader that the two indicators is not equal in general.

Example 3.1. Let $G = SL(2,\mathbb{R})$ and (π, V) be the 2-dimensional irreducible representation of G with the natural action of $SL(2,\mathbb{R})$ on a 2-dimensional vector space. It is clear that this action preserves the 2-dimensional real vector space defined by restricting scalars of V. By Definition 2.6, π is of real type, therefore $\delta(\pi) = 1$. On the other hand, the action of G preserves a skew-symmetric bilinear form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R}). \tag{1}$$

By Definition 2.4, $\varepsilon(\pi) = -1$.

Though not equal in general, the two indicators are very closely related. In most cases, the two indicators are related by a strong real form representing the real group G. See Definition 3.12 for the definition of strong real forms. The following example will hopefully provide some intuition for this relation.

Example 3.2. From basic Lie theory, we know that $G_1 = SL(2,\mathbb{R})$ and $G_2 = SU(2)$ are two different real forms of the complex Lie group $G = SL(2,\mathbb{C})$. Let π be the 2-dimensional irreducible representation of G with the natural action. Let $\pi_1 = \pi|_{G_1}$ and $\pi_2 = \pi|_{G_2}$.

It is easy to verify that Equation (1) holds for all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{C})$. This allows us to conclude $\varepsilon(\pi) = \varepsilon(\pi_1) = \varepsilon(\pi_2) = -1$. It turns out that the ε-indicator is independent of real forms in general.

On the other hand, since $SL(2)$ is compact, Theorem 2.2 implies $\delta(\pi_2) = \varepsilon(\pi_2) = -1$. We know from Example 3.1 that $\delta(\pi_1) = 1$. This computation shows that the δ-indicator is sensitive to real forms.

This crucial difference between the two indicators suggests that they might be related by the real form representing G_1. The theory of classification of real forms hints that a natural place to look is the Cartan involution of $SL(2,\mathbb{R})$:

$$\theta(g) = (g^T)^{-1}, \qquad \forall g \in SL(2,\mathbb{R}).$$

Let $x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then $\theta = \mathrm{Ad}(x)$ on $SL(2,\mathbb{R})$. Since θ is an involution of G_1, $x^2 \in Z(G_1)$. Let χ_{π_1} be the central character of π_1, we notice $\chi_{\pi_1}(x^2) = -1$. Therefore

$$\delta(\pi_1) = \chi_{\pi_1}(x^2)\varepsilon(\pi_1). \tag{2}$$

The Equation (2) is a special case of a very general fact (Theorem 3.11). The factor x represents the difference between the ordinary invariant Hermitian form (which is not necessarily positive-definite) and the c-invariant Hermitian form (which is positive-definite).

Recall $\varepsilon(\pi) = \delta(\pi)$ for π unitary. The reason for this equality is that the ordinary Hermitian form is positive-definite hence it coincides with the c-invariant Hermitian form, meaning that $\chi_\pi(x^2) = 1$.

We will give a definition for $(\mathfrak{g}_0, K)$-modules and summarize the Langlands classification. If you are familiar with $(\mathfrak{g}_0, K)$-modules and know that one can describe them using parameters on the Cartan subgroup (that's the Langlands classification), then you may wish to skip the next two subsection and come back to it later. If not, you may still want to come back when needed.

3.2 Definitions

We work within the setting of $(\mathfrak{g}_0, K)$-modules where our tools work the best and the infinite-dimensionality does not cause too much trouble.

Let $\mathfrak{g}_0$ denote the real Lie algebra of G and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ its complexification. Fix once and for all a choice of maximal compact subgroup $K \subset G$, the complexification of K is denoted $K(\mathbb{C})$.

Definition 3.2. A $(\mathfrak{g}, K)$-*module* is a pair (π, V) with V a complex vector space and π a map

$$\pi : \mathfrak{g} \cup K \to \mathrm{End}(V)$$

satisfying

1. $\pi|_\mathfrak{g}$ is a complex linear Lie algebra representation, and $\pi|_K$ is a group representation;
2. every vector $v \in V$ is K-finite, i.e. $\dim\langle \pi(K)v \rangle < \infty$
3. the differential of the action of K is equal to the restriction to $\mathfrak{k}_0$ of the action of $\mathfrak{g}$;
4. $\pi(k)(\pi(X)v) = \pi(Ad(k)X)(\pi(k)v), \quad \forall k \in K, X \in \mathfrak{g}, v \in V$

A $(\mathfrak{g}, K)$-module V is called *admissible* if all K-types are of finite multiplicity.

A *morphism* of $(\mathfrak{g}, K)$-modules is a linear map intertwining both the action of K and of $\mathfrak{g}$.

Similarly, we can define $(\mathfrak{g}_0, K)$-modules

Definition 3.3 (Definition 2.10 [2])**.** A $(\mathfrak{g}_0, K)$-*module* is a complex vector space V that is at the same time a representation of K and of the real Lie algebra $\mathfrak{g}_0$, subject to the conditions in Definition 3.2 with $\mathfrak{g}$ replaced by $\mathfrak{g}_0$.

Remark 7. A $(\mathfrak{g}_0, K)$-module (π, V) is naturally a $(\mathfrak{g}, K)$-module, with the action of $\mathfrak{g}$ on V defined as:

$$(X \otimes z) \cdot v = zX \cdot v, \quad \forall X \in \mathfrak{g}_0.$$

This extension defines an equivalence of categories from $(\mathfrak{g}_0, K)$-modules to $(\mathfrak{g}, K)$-modules.

Theorem 3.1. *[2, Corollary 2.18] Let (π, V) be a $(\mathfrak{g}, K)$-module. We can extend π to a $(\mathfrak{g}, K(\mathbb{C}))$-module such that the compatibility conditions in Definition 3.2 are satisfied. This extension defines an equivalence of categories from $(\mathfrak{g}, K)$-modules to $(\mathfrak{g}, K(\mathbb{C}))$-modules.*

Definition 3.4 (K-Finite Dual Module)**.** If (π, V) is a $(\mathfrak{g}, K)$-module, set

$$V^\vee = \{f : V \to \mathbb{C} | \dim\langle \pi^\vee(K) f\rangle < \infty \ \}$$

here $\pi^\vee(k)$ acts by the algebraic dual representation,

$$(\pi^\vee(k)f)(v) = f(\pi(k^{-1})v), \qquad \forall k \in K$$

and

$$(\pi^\vee(X)f)(v) = f(-\pi(X)v), \qquad \forall X \in \mathfrak{g}.$$

$(\pi^\vee, V^\vee)$ is called the *dual module*.

Remark 8.

1. The module $(\pi^\vee, V^\vee)$ is indeed a $(\mathfrak{g}, K)$-module, see Lemma 8.5.2 in [9]. Moreover, $(\pi^{\vee\vee}, V^{\vee\vee}) \cong (\pi, V)$ canonically.
2. Unlike the dual, "$(\mathfrak{g}, K)$-modules" does not have a canonical definition for their conjugate or Hermitian dual, neither do they have a well defined notion of the Real-Quaternionic indicator. The appropriate modules to analyze are the $(\mathfrak{g}_0, K)$-modules. We use the same definition for the *conjugate* and *Hermitian dual* as in the finite-dimensional case, see Definition 2.2 and Definition 2.3 with the word "representation" replace by $(\mathfrak{g}_0, K)$-module.

The definition of the Frobenius-Schur indicator and the Real-Quaternionic indicator is basically unchanged. Here are the corresponding definitions in the setting of $(\mathfrak{g}_0, K)$-modules.

Definition 3.5 (ε-Indicator for $(\mathfrak{g}_0, K)$-Modules)**.** Suppose (π, V) is an irreducible self-dual $(\mathfrak{g}_0, K)$-module. Then there exists bilinear form B on V that is invariant under the action of the pair $(\mathfrak{g}_0, K)$. The *Frobenius-Schur indicator* (also called the ε-indicator) is defined to be:

$$\varepsilon(\pi) = \begin{cases} 1 & B \text{ is symmetric} \\ -1 & B \text{ is skew-symmetric} \end{cases}.$$

Remark 9. The ε-indicator can be defined for $(\mathfrak{g}, K)$-modules in the same fashion.

Definition 3.6 (δ-Indicator for $(\mathfrak{g}_0, K)$-Modules)**.** Suppose (π, V) is an irreducible self-conjugate $(\mathfrak{g}_0, K)$-module. Then there exists a non-zero $(\mathfrak{g}_0, K)$ invariant conjugate-linear map $\mathcal{J} : V \to V$. For such a map, there exists $c \in \mathbb{R}^*$ such that $\mathcal{J}^2(v) = cv$ for all $v \in V$. The *Real-Quaternionic indicator* (also called the δ-indicator) is defined to be:

$$\delta(\pi) = \operatorname{sgn}(c).$$

Remark 10. To prove that the two indicators are well-defined in this setting, the most important tool is Schur's Lemma. For this reason, we want $(\mathfrak{g}_0, K)$-modules to be admissible. Since we are assuming irreducibility, by the following Theorem, we ensure that the $(\mathfrak{g}_0, K)$-modules we work with are admissible and Schur's Lemma holds for them.

Theorem 3.2. *Irreducible $(\mathfrak{g}_0, K)$-modules are admissible (Definition 3.2).*

This theorem is a classic result of Harish-Chandra's.

3.3 The Langlands Classification

We follow Section 6 of [2] closely.

Theorem 3.3. *[2, Theorem 6.1] Suppose G is a real reductive algebraic group. Then there exists a one-to-one correspondence between equivalent classes of irreducible $(\mathfrak{g}_0, K)$-modules and G-conjugacy classes of triples ("Langlands parameters")*

$$\Gamma = (H, \gamma, R^+_{i\mathbb{R}})$$

subject to the following requirements.

1. *The group H is a Cartan subgroup of G: the group of real points of a maximal torus of $G(\mathbb{C})$ defined over $\mathbb{R}$.*
2. *The character γ is level one character of the ρ_{abs} double cover of H. Write $d\gamma \in \mathfrak{h}^*$ for its differential ([2, Definition 5.1, Lemma 5.9]).*
3. *The roots $R^+_{i\mathbb{R}}$ are a positive system for the imaginary roots of H in $\mathfrak{g}$.*
4. *The weight $d\gamma$ is weakly dominant for $R^+_{i\mathbb{R}}$.*
5. *If $\alpha^\vee$ is real and $\langle d\gamma, \alpha^\vee\rangle = 0$ then $\gamma_{\mathfrak{q}}(m_\alpha) = +1$ [2, Definition 5.7].*
6. *If β is simple for $R^+_{i\mathbb{R}}$ and $\langle d\gamma, \beta^\vee\rangle = 0$ then β is non-compact.*

Two Langlands parameters are called equivalent if they are conjugate by G. Attached to each equivalent class of Langlands parameter Γ is a standard $(\mathfrak{g}_0, K)$-module $I(\Gamma)$, and it has a unique irreducible quotient module $J(\Gamma)$. The correspondence is $\Gamma \leftrightarrow J(\Gamma)$.

We will give a summary of Langlands' construction of $I(\Gamma)$. First we define some notations. Let $T = H^\theta$ be the maximal compact subgroup of H. Write $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ for the decomposition of the real Lie algebra of H into $+1$ and -1 eigenspaces of θ. Define $A = \exp(\mathfrak{a}_0)$ to be the identity component of the maximal split torus of H. Note that the group A is isomorphic to its Lie algebra and

$$H = T \times A.$$

Let $MA = \mathrm{Cent}_G(A)$ be the Langlands decomposition of the centralizer of A in G. The compact group T is a compact Cartan subgroup of M, and the parameters

$$\Lambda = (T, \gamma|_{\widetilde{T}}, R^+_{i\mathbb{R}})$$

are Harish-Chandra parameters for a limit of discrete series representation $D(\Lambda) \in \widehat{M}$. Here $\widetilde{T}$ is the $\rho_{i\mathbb{R}}$ double cover of T. For more information about this cover, see [2]. Now let

$$\nu = \gamma|_A \in \widehat{A}.$$

Choose a parabolic subgroup $P = MAN$ of G such that ν is weakly dominant for the weights of $\mathfrak{a}$ in $\mathfrak{n}$. Then the standard representation $I(\Gamma)$ can be realized as

$$I_{quo}(\Gamma) = \mathrm{Ind}_P^G(D(\Lambda) \otimes \nu \otimes 1).$$

We also use $I_{quo}(\Gamma)$ to denote the Harish-Chandra module of the standard representation. This module has a unique irreducible quotient $J(\Gamma)$. The correspondence in Theorem 3.3 is $\Gamma \leftrightarrow J(\Gamma)$.

The construction of the standard module inspires another expression of the Langlands parameter. Given Langlands parameter Γ, let

$$\Lambda = (T, \gamma|_{\widetilde{T}}, R^+_{i\mathbb{R}}),$$

and

$$\nu = \gamma|_A \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}_0, \mathbb{C}) = \mathfrak{a}_0^*.$$

Then Γ can also be written as (Λ, ν); thus $I(\Gamma) = I(\Lambda, \nu)$ and $J(\Gamma) = J(\Lambda, \nu)$. We will be using (Λ, ν) rather than Γ for the rest of this paper.

Definition 3.7. We say a character ν of A is *real* if it is a real-valued character, i.e.,

$$\nu \in \mathfrak{a}_0^*(\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{R})$$

Proposition 3.4. *Suppose $\Gamma = (\Lambda, \nu)$ is a Langlands parameter, then*

1. *The lowest K-types of $I(\Lambda, \nu)$ all have multiplicity one, and they all appear in the Langlands quotient $J(\Lambda, \nu)$. ([2, page 78], [9, Theorem 4.3.16])*

2. *The infinitesimal character of $J(\Lambda, \nu)$ is real [9, Definition 5.4.11] if and only if $\nu \in \mathfrak{a}_0^*$ is real (Definition 3.7).*

For the main part of this paper, we will assume real infinitesimal character. In the next subsection we will talk about different Hermitian forms, including the ordinary invariant Hermitian forms and the c-invariant Hermitian forms, and their relations.

3.4 Hermitian Forms

The relation between the ε-indicator and the δ-indicator fundamentally depends on the relation between the ordinary invariant Hermitian form and the c-invariant Hermitian form. In this section, we introduce these notions.

It is more convenient to work in the setting of $(\mathfrak{g}, K(\mathbb{C}))$-modules instead of $(\mathfrak{g}_0, K)$-modules. So (π, V) will denote a $(\mathfrak{g}, K(\mathbb{C}))$-module in this subsection.

We will assume some basic knowledge of the classification of real forms. For an introduction to the theory of real forms, one can refer to [6] and [2] for a introduction of real forms.

Given a real form σ_0 of $G(\mathbb{C})$ corresponding to G, there is always a compact form σ_c, unique up to conjugation by G. One can always choose σ_c such that σ_0 and σ_c commute. The involution $\theta = \sigma_0 \circ \sigma_c$ is the Cartan involution of G. The involutions σ_0 and σ_c are natual real structures on the pair $(\mathfrak{g}, K(\mathbb{C}))$. The notion of Hermitian form depends on the choice of a real structure on $G(\mathbb{C})$ (for details of real structure see [2, Section 8]). The usual real structure associated to σ_0 defines an ordinary Hermitian form; and σ_c defines a c-invariant Hermitian form.

Recall the definition of Hermitian transpose.

Definition 3.8. If $T \in \mathrm{Hom}(V, W)$, then the *Hermitian transpose* of T is

$$T^h \in \mathrm{Hom}(W^h, V^h), \qquad T^h(\xi)(v) = \xi(T(v)) \qquad v \in V, \xi \in W^h.$$

Definition 3.9. Suppose (π, V) is a $(\mathfrak{g}, K(\mathbb{C}))$-module and σ is a real structure on the pair $(\mathfrak{g}, K(\mathbb{C}))$ ([2, Definition 8.1]). The *σ-Hermitian dual* of (π, V) is denoted $(\pi^{h,\sigma}, V^h)$ with

$$\pi^{h,\sigma} : K \to GL(V^h), \qquad \pi^{h,\sigma}(k) = [\pi(\sigma(k^{-1})]^h, \qquad \forall k \in K(\mathbb{C})$$

$$\pi^{h,\sigma} : \mathfrak{g} \to \mathrm{End}(V^h), \qquad \pi^{h,\sigma}(X) = -[\pi(\sigma(X))]^h, \qquad \forall X \in \mathfrak{g}.$$

The operator $\pi(k)^h$ is the Hermitian transpose of the operator $\pi(k)$, similarly for $\pi(X)^h$.

Definition 3.10. [2, Definition 8.6] Suppose (π, V) is a $(\mathfrak{g}, K(\mathbb{C}))$-module, a *$\sigma$-Hermitian form* on V is a Hermitian pairing $\langle , \rangle^\sigma$ on V satisfying

$$\langle X \cdot v, w\rangle^\sigma = \langle v, -\sigma(X) \cdot w\rangle^\sigma \qquad \forall X \in \mathfrak{g}, v, w \in V$$

$$\langle k \cdot v, w\rangle^\sigma = \langle v, \sigma(k^{-1}) \cdot w\rangle^\sigma \qquad \forall k \in K(\mathbb{C}), v, w \in V$$

such a form may be identified with an intertwining operator

$$T \in \mathrm{Hom}_{\mathfrak{g},K}(\pi, \pi^{h,\sigma})$$

with the Hermitian condition: $T = T^h$

Remark 11. A $(\mathfrak{g}_0, K)$-module π is said to be Hermitian if and only if it admits a σ_0-Hermitian form.

Lemma 3.5.

1. *A $(\mathfrak{g}, K(\mathbb{C}))$-module (π, V) has a σ-Hermitian form if and only if $\pi \cong \pi^{h,\sigma}$.*
2. *A $(\mathfrak{g}_0, K)$-module (π, V) is Hermitian if and only if its extended $(\mathfrak{g}, K(\mathbb{C}))$-module is σ_0-Hermitian.*

In this paper, we are particularly interested in $\langle , \rangle^0$ and $\langle , \rangle^c$. We often call the former the "ordinary Hermitian form" and the latter "c-invariant Hermitian form". It is the focus of this section to establish the connection between the two invariant Hermitian forms. In order to do that, we first introduce the notion of "extended pairs".

Definition 3.11. [2, Definition 8.12] Let μ be an automorphism of the pair $(\mathfrak{g}, K(\mathbb{C}))$ with the property that $\mu^2 = \mathrm{Ad}(\lambda), \ \ \lambda \in K(\mathbb{C})$ is an inner automorphism of the pair $(\mathfrak{g}, K(\mathbb{C}))$. The corresponding *extended group* is the extension

$$1 \to K(\mathbb{C}) \to {}^\mu K(\mathbb{C}) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

with a specified generator μ_1 mapping to $1 \in \mathbb{Z}/2\mathbb{Z}$, and subject to the relation $\mu_1 k \mu_1^{-1} = \mu(k)$ for $k \in K(\mathbb{C})$, and $\mu_1^2 = \lambda$. This extended group acts by automorphisms on $\mathfrak{g}$, and we call $(\mathfrak{g}, {}^\mu K(\mathbb{C}))$ an *extended pair.*

Now we establish the connection between different σ-Hermitian forms. The next Proposition lays the foundation of proving the main theorem of this paper.

Proposition 3.6. *Suppose (π, V) is an irreducible $(\mathfrak{g}, K(\mathbb{C}))$-module, and V has a non-degenerate σ_c-Hermitian form $\langle , \rangle^{\sigma_c}$, unique up to a real multiple. Then:*

1. *The following are equivalent:*

   (a) *There is a non-degenerate σ_0-Hermitian form $\langle,\rangle^{\sigma_0}$. It is unique up to a real multiple.*

   (b) *(π, V) is isomorphic to its twist by θ:*

$$D : (\pi, V) \cong (\pi^{\theta}, V)$$

   In this case D is unique up to a complex multiple.

   Assuming $\theta^2 = Ad(\lambda)$ for some $\lambda \in K(\mathbb{C})$, then (a) and (b) are also equivalent to the following.

   (c) *The $(\mathfrak{g}, K(\mathbb{C}))$-module V extends to a $(\mathfrak{g}, {}^{\theta}K(\mathbb{C}))$-module with θ acting as D subject to the additional requirement that $D^2 = \pi(\lambda)$. If any such extensions exist then there are exactly two. The operators D differ by multiplication by ± 1.*

Assuming from now on the condition of (c) is satisfied, and the operator D is chosen to be that in (c). Furthermore, assume $\langle \lambda \cdot v, \lambda \cdot w\rangle^{\sigma_c} = \langle v, w\rangle^{\sigma_c}$.

2 *There is a non-zero complex number ξ so that*

$$D^{-1} = \xi D$$

on V, and $|\xi|^2 = 1$

3 *The form $\langle v, w\rangle' = \langle D(v), D(w)\rangle^{\sigma_c}$ is again a σ_c-Hermitian form, so there exists a real number ω such that*

$$\langle D(v), D(w)\rangle^{\sigma_c} = \omega \langle v, w\rangle^{\sigma_c}$$

4 *The scalar ω is ± 1.*

5 *If ζ is a square root of $\omega\bar{\xi}$ Then*

$$\langle v, w\rangle^{\sigma_0} := \zeta^{-1} \langle D(v), w\rangle^{\sigma_c} = \zeta \langle v, D(w)\rangle^{\sigma_c}$$

is a σ_0-Hermitian form on V.

Remark 12. The proof of this Proposition is straightforward but somewhat tedious. Interested reader can find detailed proof of this Proposition in the author's thesis. It is also a special case of [2, Proposition 8.9].

This Proposition allows us to write downs a precise equation relating $\langle,\rangle^0$ and $\langle,\rangle^c$ whenever they both exist. It turns out the operator D is related to a strong real form (Definition 3.12) of G. If G is equal rank, this x lives in G. If G is unequal rank, then it lives in an extended group of G. The distinction affects our technique in some of the proofs. So we will discuss the two cases separately.

It is also useful to see the action of σ-Hermitian dual on Langlands parameters.

Theorem 3.7. *Suppose $\Gamma = (\Lambda, \nu)$ is the Langlands parameter for the $(\mathfrak{g}_0, K)$-module $(\pi(\Gamma), J(\Gamma))$.*

1. *The σ_c-Hermitian dual of this module has the Langlands parameter*

$$\Gamma^{h,\sigma_c} = (\Lambda, \nu)^{h,\sigma_c} = (\Lambda, \overline{\nu}).$$

In particular, if ν is real, then $\Gamma = \Gamma^{h,\sigma_c}$.

2. *The σ_0-Hermitian dual of this module has the Langlands parameter*

$$\Gamma^{h,\sigma_0} = (\Lambda, \nu)^{h,\sigma_0} = (\Lambda, -\overline{\nu}).$$

Proposition 3.8. *[2, Proposition 10.7] Suppose $\Gamma = (\Lambda, \nu)$ is a Langlands parameter for G, and $\Gamma^{h,\sigma_c} = (\Lambda, \overline{\nu})$ is the c-Hermitian dual parameter, then*

1. *The irreducible quotient module $J(\Gamma)$ admits a c-invariant Hermitian form if and only if Γ is equivalent to Γ^{h,σ_c}.*
2. *$J(\Lambda, \nu)$ admits a c-invariant Hermitian form if and only if there exists $w \in W(G, H)$, where H is a real Cartan subgroup of G, such that $w \cdot \Lambda = \Lambda$ and $w \cdot \nu = \overline{\nu}$. In particular, if ν is real then $J(\Lambda, \nu)$ has a c-form.*
3. *Suppose ν is real, then any c-invariant Hermitian form on $J(\Lambda, \nu)$ has the same sign on the lowest K-types. In particular, the form can be chosen to be positive-definite on every lowest K-type.*

This Proposition shows that by restricting to $(\mathfrak{g}_0, K)$-modules with real infinitesimal character, we can always assume the existence of c-invariant Hermitian form. It is worth pointing out that this condition is quite weak. For instance, when G is semi-simple or when $Z(G)$ contains no split torus, then every finite-dimensional representation is of real infinitesimal character. Also every unitary representation can be unitarily induced from a unitary representation with real infinitesimal character.

3.5 Modules of Equal Rank Groups

To remind the reader, we talked about why we should split the equal rank and unequal rank cases in Remark 12. In this subsection, G will denote a real reductive algebraic group which is also equal rank (Definition 3.12). The $(\mathfrak{g}_0, K)$-modules we are interested in are irreducible, self-conjugate (so that the δ-indicator is well defined), and of real infinitesimal character (so that the c-invariant Hermitian form exists).

Definition 3.12 (Equal Rank, Unequal Rank, Strong Real Form)**.** Suppose G is a real reductive algebraic group and $K = G^\theta$ is a maximal compact subgroup, then G is said to be *equal rank* if the rank of G is equal to the rank of K. Equivalently, the automorphism θ of G is inner. In this case, a *strong real form* for G is an element $x \in G$ such that $\mathrm{Ad}(x) = \theta$. The definition of *unequal rank* groups is the natural negation of the above conditions.

Remark 13. If so then $K = \mathrm{Cent}_G(x)$, so $x \in Z(K)$, and $x^2 \in Z(K)$. It also follows that $x^2 = z \in Z(G) \cap K$.

Lemma 3.9. *If G is equal rank, and ν is real, then $J(\Lambda, \nu)$ is Hermitian and there exists an invariant Hermitian form.*

Proof. Since $\theta = \sigma_0 \circ \sigma_c$, we know:

$$[J(\Lambda, \nu)^{h,\sigma_c}]^\theta \cong J(\Lambda, \nu)^{h,\sigma_0}.$$

The group G being equal rank implies that θ is inner. Therefore

$$J(\Lambda, \nu)^{h,\sigma_c} \cong J(\Lambda, \nu)^{h,\sigma_0}.$$

By Proposition 3.8, ν real implies

$$J(\Lambda, \nu) \cong J(\Lambda, \overline{\nu}) \cong J(\Lambda, \nu)^{h,\sigma_c}.$$

Therefore

$$J(\Lambda, \nu) \cong J(\Lambda, \nu)^{h,\sigma_c} \cong J(\Lambda, \nu)^{h,\sigma_0}$$

i.e., $J(\Lambda, \nu)$ is Hermitian. □

Proposition 3.6 allows us to define an invariant Hermitian form based on the existing c-invariant Hermitian form.

Lemma 3.10. *Let G be an equal rank real reductive algebraic group and (π, V) be an irreducible $(\mathfrak{g}_0, K)$-module which is both self-conjugate and of real infinitesimal character. Then the σ_0-Hermitian form is related to the σ_c-Hermitian form by*

$$\langle v, w\rangle^{\sigma_0} = \zeta^{-1}\langle x \cdot v, w\rangle^{\sigma_c}$$

where x is a strong real form for G, $z := x^2 \in Z(K)$, and ζ is a square root of $\chi_\pi(z)$. Note that ζ is independent of v and w.

This form is non-degenerate and σ_0-invariant. In particular, $\langle , \rangle^{\sigma_0}$ is an ordinary invariant Hermitian form under the actions of $(\mathfrak{g}_0, K)$.

We are now ready for the first main theorem of this paper.

Theorem 3.11. *Let G be an equal rank real reductive algebraic group and (π, V) be an irreducible $(\mathfrak{g}_0, K)$-module which is both self-conjugate and of real infinitesimal character. Then*

$$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2),$$

where $x \in K$ is a strong real form of G.

Proof. Since π is self-conjugate and Hermitian (Lemma 3.9), π is also self-dual. Therefore admits a non-degenerate invariant bilinear form B. Let $\langle , \rangle^0$ be the form defined in Lemma 3.10. Define map $\mathcal{J} : V \to V$ such that

$$B(v, w) = \langle v, \mathcal{J}(w)\rangle^0, \qquad \forall v, w \in V.$$

It is easy to see that $\mathcal{J}$ is conjugate linear, non-zero and $(\mathfrak{g}_0, K)$-equivariant. Recall the computation in the proof of Theorem 2.2, we replace $\langle , \rangle$ with the Hermitian form $\langle , \rangle^0$ and get:

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle^0}{\langle w, v\rangle^0}\right), \qquad \forall v, w \in V \text{ where } \langle v, w\rangle \neq 0.$$

Rewriting this equation using the c-invariant Hermitian form $\langle , \rangle^c$ we have:

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w)\rangle^0}{\overline{\langle v, w\rangle^0}}\right) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle x \cdot \mathcal{J}(v), \mathcal{J}(w)\rangle^c}{\overline{\zeta^{-1}\langle x \cdot v, w\rangle^c}}\right).$$

Since $x \in K$ by Definition 3.12, $\mathcal{J}$ is equivariant under the action of x, i.e., $x \cdot \mathcal{J}(v) = \mathcal{J}(x \cdot v)$. Set $w = x \cdot v$ to be an element in the lowest K-types of π, then

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\zeta^{-1}\langle w, w\rangle^c}}\right) = \operatorname{sgn}(\zeta^{-1}\overline{\zeta})\operatorname{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w, w\rangle^c}}\right).$$

Because ζ is a square root of a central character for a self-dual representation, it's not hard to see that $|\zeta| = 1$ hence $\overline{\zeta} = \zeta^{-1}$ and $\operatorname{sgn}(\zeta^{-1}\overline{\zeta}) = \operatorname{sgn}(\zeta^{-2}) = \chi_\pi(z^{-1}) = \chi_\pi(x^{-2})$.

It remains to determine

$$\operatorname{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w, w\rangle^c}}\right).$$

Suppose w is in the lowest K-type with highest weight λ. Then $\mathcal{J}(w)$ is in the $\overline{\lambda}$ weight space. The weight $\overline{\lambda}$ of the same length as λ, therefore $\overline{\lambda}$ again represents a lowest K-type. So Proposition 3.8(3) implies that $\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c > 0$ and $\langle w, w\rangle^c > 0$. So

$$\operatorname{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w, w\rangle^c}}\right) = 1$$

and

$$\varepsilon(\pi)\delta(\pi) = \chi_\pi(x^{-2}) \Rightarrow \delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$$

as desired. □

Remark 14. There are many choices for a strong real form x of G. The formula in Theorem 3.11 is independent of the choice of x. Suppose y is another strong real form of G, then $y = zx$ for some $z \in Z(G)$. The central character χ_π evaluted on x^2 equals that evaluated on y^2 because $\chi_\pi(z^2) = 1$ for any $z \in Z(G)$. This is essentially because π is self-dual. This argument can be repeated for the rest of the Theorems in this paper, whenever a strong real form is involved.

3.6 Modules of Unequal Rank Groups

In this subsection, G denotes an unequal rank real reductive algebraic group, and (π, V) is an irreducible, self-conjugate $(\mathfrak{g}_0, K)$-module with real infinitesimal character.

Since G is unequal rank, the Cartan involution of G is no longer inner. That means x does not live in G (or K) anymore, but it is an element of an extended group of G.

3.6.1 Extended Group

Definition 3.13. A *splitting datum* or *pinning* is a set $\mathcal{P} = \{B, H, \{X_\alpha\}$ where B is a Borel subgroup, H is a Cartan subgroup contained in B and $\{X_\alpha\}$ is a set of root vectors for the simple roots of H in B.

Definition 3.14. An involution of G is said to be *distinguished* if it preserves a pinning.

For G and the Cartan involution θ of G, we can define the distinguished involution γ in the inner class of θ canonically. The interested reader can find how to define γ in [2, Section 12]. The γ defined this way has the properties:

$$\gamma^2 = 1, \quad \gamma \circ \theta = \theta \circ \gamma, \quad \gamma \circ \sigma_c = \sigma_c \circ \gamma. \tag{3}$$

Definition 3.15. The *extended group* ${}^{\gamma}G(\mathbb{C})$ for $G(\mathbb{C})$ is the semi-direct product

$${}^{\gamma}G(\mathbb{C}) = G(\mathbb{C}) \rtimes \{1, \gamma\}$$

According to (3), γ preserves G, K, $K(\mathbb{C})$. We can therefore define all the corresponding extended groups ${}^{\gamma}G$, ${}^{\gamma}K$, and ${}^{\gamma}K(\mathbb{C})$.

Definition 3.16. A strong real form x of G satisfies is an element in ${}^{\gamma}G \backslash G$ such that $\mathrm{Ad}(x) = \theta$ on G. In this case, $x = x_0\gamma$, $x_0 \in K$ and $x_0^2 = x^2 \in Z(K)$.

A consequence of θ being outer is that π is not necessarily Hermitian if it has real infinitesimal character as stated in Lemma 3.9.

Lemma 3.12. *Assume G and π satisfies all conditions stated in the beginning of Section 3.6. Then π is Hermitian if and only if $\pi \cong \pi^{\gamma}$.*

We treat the Hermitian case and non-Hermitian case separately.

3.6.2 Hermitian Modules

In addition to the conditions we put on π in the beginning of Section 3.6, we also assume that π is Hermitian. Consequently, $\pi \cong \pi^{\gamma}$ by Lemma 3.12. By Clifford theory and more specifically Proposition 3.6, π extends to two $(\mathfrak{g}_0, {}^{\gamma}K)$-modules, denoted π_1 and π_2. They satisfy $\pi_1(\gamma) = -\pi_2(\gamma)$. We can write down a similar equation as in Lemma 3.10 with x in the extended group. The extensions π_1 and π_2 gives meaning to the expression $x \cdot v$ below.

Lemma 3.13. *Let G be an unequal rank real reductive algebraic group and (π, V) be an irreducible $(\mathfrak{g}_0, K)$-module which is self-conjugate, Hermitian, and of real infinitesimal character. Then the σ_0-Hermitian form is related to the σ_c-Hermitian form by*

$$\langle v, w \rangle^{\sigma_0} = \zeta^{-1} \langle x \cdot v, w \rangle^{\sigma_c}$$

where x is a strong real form for G, $x \in {}^{\gamma}K$, $z := x^2 \in Z(K)$, and ζ is a square root of $\chi_\pi(z)$. Note that ζ is independent of v and w.

This form is non-degenerate and σ_0-invariant. In particular, $\langle , \rangle^{\sigma_0}$ is an ordinary invariant Hermitian form under the actions of $(\mathfrak{g}_0, K)$.

Proof. We have already established that there exists a σ_c-invariant Hermitian form for (π, V) under these assumptions. The rest of the proposition is simply a special case of Proposition 3.6, taking D to be $\pi(x)$ and $\lambda = x^2 = z \in Z(K)$. In this case $\omega = 1$, $\xi = \zeta^{-2}$ and the square root of $\omega\overline{\xi} = \xi^{-1}$ is ζ.

We are downplaying the distinction between $(\mathfrak{g}_0, K)$-modules and $(\mathfrak{g}, K(\mathbb{C}))$-modules, because the difference is not of essential importance to us and it may be a potential distraction. $\square$

Lemma 3.14. *Let G and (π, V) satisfy the conditions given in the beginning of Section 3.6.2. Then the extensions π_1 and π_2 both admit c-invariant Hermitian forms and are Hermitian.*

Proof. To say π_1 admits a c-Hermitian form is equivalent to saying $\pi_1 \cong [\pi_1]^{h,\sigma_c}$. In fact, the module $[\pi_1]^{h,\sigma_c}$ is also an extension of π. Because there are only two extensions of π, we will instead prove

$$[\pi_1]^{h,\sigma_c} \ncong \pi_2.$$

Suppose there exists such an isomorphism:

$$\psi : \pi_2 \to [\pi_1]^{h,\sigma_c}$$

$$[\pi_1]^{h,\sigma_c}(\gamma) \cdot \psi(v) = \psi(\pi_2(\gamma)v). \tag{4}$$

Let $v \in V$ such that $\pi_1(\gamma)v = cv$ some $c \in \mathbb{C}^*$. Such vector exists because γ acts as an isomorphism $D : \pi \to \pi^\gamma$ and D is of finite order.

By definition, $\pi_2(\gamma)v = -cv$. We also know that $\psi(v)(v) \neq 0$ because the c-Hermitian form on V is positive definite. Equation (4) left hand side evaluated on v is

$$[\pi_1]^{h,\sigma_c}(\gamma) \cdot \psi(v)(v) = \psi(v)(\pi_1(\gamma^{-1})v) = \psi(v)(c^{-1}v) = \overline{c^{-1}}\psi(v)(v)$$

The right hand side evaluated on v equals to $-c\psi(v)(v)$. The two sides equal if and only if $-c = \overline{c^{-1}}$ if and only if $|c| = -1$. This contradicts the fact that $|c| \geq 0$ for all $c \in \mathbb{C}$. Therefore $\pi_1 \cong \pi_1^{h,\sigma_c}$.

It remains to show that the extensions are Hermitian. This is immediate because $\theta = \sigma_c \circ \sigma_0$ and γ is inner to θ. ☐

Definition 3.17. Let G and π satisfy the conditions given in the beginning of Section 3.6.2. Suppose π_1 is an extended $(\mathfrak{g}_0, {}^\gamma K)$-module of π. Define $\kappa(\pi)$ as follows:

$$\kappa(\pi) = \begin{cases} 1 & \pi_1 \text{ is self-dual} \\ -1 & \pi_1 \text{ is not self-dual} \end{cases}.$$

Remark 15. Note that κ only depends on π, even though it uses the definition of π_1.

Theorem 3.15. *Let G be an unequal rank real reductive algebraic group and (π, V) be an irreducible self-conjugate, Hermitian $(\mathfrak{g}_0, K)$-module with real infinitesimal character. Then we have the following equation:*

$$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)\kappa(\pi)$$

where $x \in {}^\gamma K$ is a strong real form given by G.

Proof. From previous calculations, we have:

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle x \cdot \mathcal{J}(v), \mathcal{J}(w)\rangle^c}{\overline{\zeta^{-1}\langle x \cdot v, w\rangle^c}}\right) = \operatorname{sgn}(\zeta^{-2})\operatorname{sgn}\left(\frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w)\rangle^c}{\overline{\langle x \cdot v, w\rangle^c}}\right). \tag{5}$$

As before we would like to have x commutes with $\mathcal{J}$. Here, it is not a given, because x is in the extended group.

If the extended module π_1 is self-dual, then B is invariant under the action of x. Therefore

$$\langle v, \mathcal{J}(x \cdot w)\rangle^{\sigma_0} = B(v, x \cdot w) = B(x^{-1} \cdot v, w) = \langle x^{-1} \cdot v, \mathcal{J}(w)\rangle^{\sigma_0} = \langle v, x \cdot \mathcal{J}(w)\rangle^{\sigma_0}$$

implies

$$\mathcal{J}(x \cdot v) = x \cdot \mathcal{J}(v) \quad \forall v \in V$$

If the extended module π_1 is not self-dual, then $\pi_1^\vee \cong \pi_2$ since $\pi_1^\vee$ is easily proven to be an extension of π. We can take the isomorphism $\psi : \pi_2 \cong \pi_1^\vee$ and it will serve as an $(\mathfrak{g}_0, K)$-invariant bilinear form on V:

$$B(v, w) = \psi(v)(w) \quad \forall v, w \in V.$$

The map ψ being an intertwiner means

$$\psi(\pi_2(x)v) = \pi_1^\vee(x)\psi(v).$$

This together with the fact that $x = x_1\gamma$ implies

$$\begin{aligned} B(x \cdot v, w) = B(\pi_1(x)v, w) &= \psi(\pi_1(x)v)(w) = \psi(-\pi_2(x)v)(w) \\ &= -\pi_1^\vee(x)\psi(v)(w) = -\psi(v)(\pi_1(x^{-1})w) = -B(v, x \cdot w) \end{aligned}$$

We define an index κ that takes the value 1 if π_1 is self-dual and -1 if π_1 is not self-dual. Then

$$B(x \cdot v, w) = \kappa(\pi)B(v, x^{-1} \cdot w)$$

This implies

$$x \cdot \mathcal{J}(v) = \kappa(\pi)\mathcal{J}(x \cdot v)$$

Equation (5) hence becomes

$$\begin{aligned} \varepsilon(\pi)\delta(\pi) &= \operatorname{sgn}(\zeta^{-2})\operatorname{sgn}\left(\frac{\langle \kappa(\pi)\mathcal{J}(x \cdot v), \mathcal{J}(w)\rangle^c}{\overline{\langle x \cdot v, w\rangle^c}}\right) \\ &= \chi_\pi(x^2)\kappa(\pi)\operatorname{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w, w\rangle^c}}\right) \end{aligned}$$

By the proof of Theorem 3.11, $\mathcal{J}$ sends lowest K-types to lowest K-types. Since the c-invariant Hermitian form is positive-definite on all lowest K-types, the theorem easily follows. □

Remark 16. The index $\kappa(\pi) = 1$ for finite-dimensional π. The proof of this fact is in the author's thesis.

3.6.3 Non-Hermitian Modules

We assume for G and (π, V) the conditions stated in the beginning of Section 3.6. In addition, we assume π is not Hermitian. By Lemma 3.12, we know that $\pi \ncong \pi^\gamma$. Clifford theory implies that we can induce π to an irreducible $(\mathfrak{g}_0, {}^\gamma K)$-module, denoted $\widetilde{\pi}$.

The following lemmas are needed for the proof of the main theorem (Theorem 3.22) of this section.

Lemma 3.16. *If π has real infinitesimal character then $\widetilde{\pi}$ has real infinitesimal character.*

Proof. The module $\widetilde{\pi}$ restricted to $(\mathfrak{g}_0, K)$ splits into two modules $\widetilde{\pi} = \pi + \pi^\gamma$. The action of $\mathfrak{a}_0$ on J is by the real valued character ν. We will show that ν^γ is again real valued. By definition θ acts on $\mathfrak{a}_0$ by -1 and γ is inner to θ, i.e., $\theta = \mathrm{Ad}(x_0) \circ \gamma$ where $x_0 \in H_f$ by the discussion in [2, P80]. Therefore

$$\nu^\gamma(X) = \nu(\gamma(X)) = \nu(x_0\theta(X)x_0^{-1}) = \nu(-X) = -\nu(X).$$

Clearly ν^γ is real valued on $\mathfrak{a}_0$. □

Lemma 3.17. *[2, Proposition 12.7] Suppose $\widetilde{\pi}$ is an irreducible $(\mathfrak{g}_0, {}^\gamma K)$-module of real infinitesimal character. Then $\widetilde{\pi}$ admits a non-degenerate c-invariant Hermitian form that is unique up to a real scalar multiple. It can be chosen to be positive-definite on the lowest ${}^\gamma K$-types of $\widetilde{\pi}$.*

Lemma 3.18. *The extended module $\widetilde{\pi}$ is Hermitian and self-dual.*

Proof. The twist of $\widetilde{\pi}$ is $\mathrm{Ind}\pi^\gamma$ therefore isomorphic to $\widetilde{\pi}$. By Proposition 3.17 $\widetilde{\pi} \cong \widetilde{\pi}^{h,\sigma_c}$. Therefore $\widetilde{\pi} \cong \widetilde{\pi}^{h,\sigma_0}$.

The module $\widetilde{\pi}$ is self-conjugate given that π is self-conjugate. It is a consequence that $\widetilde{\pi}$ is self-dual. □

Lemma 3.19. *Suppose (π, V) is an irreducible $(\mathfrak{g}_0, K)$-module and $(\widetilde{\pi}, \widetilde{V})$ its irreducibly induced $(\mathfrak{g}_0, {}^\gamma K)$-module. If $\delta(\pi)$ and $\delta(\widetilde{\pi})$ both exist, then $\delta(\pi) = \delta(\widetilde{\pi})$.*

Proof. Since π is self-conjugate, there exists $\mathcal{J} : V \to V$ conjugate linear and $(\mathfrak{g}_0, K)$ equivariant. By definition of induced representation $\widetilde{V} = V \oplus \gamma V$. Define $\widetilde{\mathcal{J}} : \widetilde{V} \to \widetilde{V}$ such that:

$$\widetilde{\mathcal{J}}(v + \gamma w) = \mathcal{J}(v) + \gamma\mathcal{J}(w), \quad \forall v \in V, \gamma w \in \gamma V$$

It is easy to see that $\widetilde{\mathcal{J}}$ is conjugate linear and ${}^\gamma G$-invariant. We will demonstrate the calculation for ${}^\gamma G$-invariance. For $g \in G$:

$$\begin{aligned}\widetilde{\mathcal{J}}(g \cdot (v + \gamma w)) &= \widetilde{\mathcal{J}}(g \cdot v + \gamma\gamma(g) \cdot w) = \mathcal{J}(g \cdot v) + \gamma\mathcal{J}(\gamma(g) \cdot w)\\ &= g \cdot \mathcal{J}(v) + \gamma\gamma(g) \cdot \mathcal{J}(w) = g \cdot \mathcal{J}(v) + g \cdot \gamma\mathcal{J}(w)\\ &= g \cdot \widetilde{\mathcal{J}}(v + \gamma w)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathcal{J}}(\gamma \cdot (v + \gamma w)) &= \widetilde{\mathcal{J}}(\gamma v + zw) = \gamma\mathcal{J}(v) + z\mathcal{J}(w) = \gamma(\mathcal{J}(v) + \gamma\mathcal{J}(w))\\ &= \gamma\widetilde{\mathcal{J}}(v + \gamma w)\end{aligned}$$

Then for $v \in V$:

$$\delta(\pi)v = \mathcal{J}^2(v) = \widetilde{\mathcal{J}}^2(v) = \delta(\widetilde{\pi})v$$

□

Proposition 3.20. *Assume G and $\widetilde{\pi}$ satisfy the conditions in the beginning of this section. Fix a strong real form $x \in {}^\gamma K \backslash K$ such that $x^2 \in Z(K)$ acts on π as a scalar ζ^2. Therefore the central element x^2 also acts on π^γ by ζ^2. Fix* *a c-invariant Hermitian form* $\langle , \rangle^c$ *on* $\widetilde{\pi}$ *which is positive-definite on the lowest* ${}^\gamma K$*-types. If* $\widetilde{\pi}$ *admits an invariant Hermitian form, then we can define* $\langle , \rangle^0$

$$\langle \widetilde{v}, \widetilde{w} \rangle^0 = \zeta^{-1} \langle x \cdot \widetilde{v}, \widetilde{w} \rangle^c = \zeta \langle \widetilde{v}, x \cdot \widetilde{w} \rangle^c$$

and it is a $(\mathfrak{g}_0, {}^\gamma K)$*-invariant Hermitian form on* $\widetilde{\pi}$*.*

Proof. This is again a consequence of Proposition 3.6 with λ replaced by x. The verification of $\langle , \rangle^0$ being an ordinary invariant Hermitian form is elementary. □

Now we have shown that all the good properties that we want and π does not have are possessed by $\widetilde{\pi}$. Also $\widetilde{\pi}$ inherited the good properties of π. For example, having a c-invariant Hermitian form etc. This enables us to use the previous arguments for Hermitian modules of equal rank groups on $\widetilde{\pi}$ to obtain a formula for $\delta(\widetilde{\pi})$.

Theorem 3.21. *Suppose* $(\widetilde{\pi}, \widetilde{V})$ *is the the induced module of* (π, V) *with* π *satisfying all the conditions we set in this section. Then*

$$\delta(\widetilde{\pi}) = \varepsilon(\widetilde{\pi}) \chi_{\widetilde{\pi}}(x^2).$$

Because of the above lemmas, the proof of this theorem can be obtained by replacing π with $\widetilde{\pi}$ in the proof of Theorem 3.11.

Let's come back to our main subject of interest here: $\delta(\pi)$. Lemma 3.19 shows that $\delta(\pi) = \delta(\widetilde{\pi})$; the proof applies in the infinite-dimensional case.

Theorem 3.22. *Let* G *be a real reductive algebraic group which is unequal rank,* (π, V) *be an irreducible* $(\mathfrak{g}_0, K)$*-module that is not Hermitian but self-conjugate and of real infinitesimal character. Then*

$$\delta(\pi) = \varepsilon(\widetilde{\pi}) \chi_{\pi}(x^2)$$

where $\widetilde{\pi} = Ind_{(\mathfrak{g}_0, K)}^{(\mathfrak{g}_0, {}^\gamma K)} \pi$ *and* $x \in {}^\gamma K \backslash K$ *is a strong real form of* G*.*

This theorem is a direct corollary of Theorem 3.21 and Lemma 3.19, with the additional observation that $\chi_\pi(x^2) = \chi_{\widetilde{\pi}}(x^2)$.

4 Formula for Finite-Dimensional Representations

The finite-dimensional case is comparatively more clean and can be useful to some readers. Here we give one theorem for the relation of $\varepsilon(\pi)$ and $\delta(\pi)$ for π finite-dimensional. Then we present a closed formula for the δ-indicator of finite-dimensional representations.

Theorem 4.1. *Let* G *be a real reductive algebraic group, and* π *be an irreducible, finite-dimensional, and self-conjugate representation of* G *with real infinitesimal character. Then*

1. *If* π *is Hermitian, then* $\delta(\pi) = \varepsilon(\pi) \chi_\pi(x^2)$*;*
2. *if* π *is non-Hermitian, then* $\delta(\pi) = \varepsilon(\widetilde{\pi}) \chi_\pi(x^2)$*.*

Here $\widetilde{\pi} = Ind_G^{\gamma G}\pi$, x *is a strong real form of* G.

Remark 17. The indicator $\varepsilon(\widetilde{\pi})$ is understood when G is simple, in that case, the Chevalley involution is either trivial or inner to γ. The formula for $\varepsilon(\widetilde{\pi})$ is given in [1].

The Frobenius-Schur indicator is given in terms of the central character in Bourbaki.

Theorem 4.2. *[3, Ch.IX, 7.2, Proposition 1] Let* $G(\mathbb{C})$ *be a connected complex reductive Lie group,* π *an irreducible finite-dimensional self-dual representation of* $G(\mathbb{C})$*. Then*

$$\varepsilon(\pi) = \chi_\pi(z(\rho^\vee)).$$

Here $z(\rho^\vee) = \exp(2\pi i \rho^\vee)$ *and* $\rho^\vee$ *is the half sum of the positive co-roots.*

For proof, see [1, Lemma 5.2]. If we replace $G(\mathbb{C})$ by a real form G, the same theorem holds.

Theorem 4.3. *Suppose* G *is a real reductive algebraic group, and* π *is an irreducible finite-dimensional self-conjugate Hermitian representation of* G *with real infinitesimal characters. Then*

$$\delta(\pi) = \chi_\pi(z(\rho^\vee) \cdot x^2),$$

where x *is a strong real form of* G.

Note that if G is semi-simple, then π is finite-dimensional implies that π has real infinitesimal character. Therefore Theorem 4.3 can be further simplified under the assumtion that G is semi-simple.

Theorem 4.4. *Suppose* G *is a semi-simple algebraic group, and* π *is an irreducible finite-dimensional self-conjugate Hermitian representation of* G*. Then*

$$\delta(\pi) = \chi_\pi(z(\rho^\vee) \cdot x^2),$$

where x *is a strong real form of* G.

5 Errata for [6]

The formula in Theorem 4.3 enables us to quickly calculate the δ-indicator for finite-dimensional representations for simple Lie algebras. Hence we were able to correct mistakes in the table of indicators in [6, page 292].

The notation $\mathfrak{g}$ here denotes a real form of a simple complex Lie algebra. $\rho(\Lambda)$ is an irreducible complex representation of $\mathfrak{g}$ with highest weight Λ such that $\overline{\rho} \cong \rho$. A well known fact is that Λ can be written as a linear combination of fundamental representations with coefficient of the i^{th} fundamental representation Λ_i.

The asterisks mark the corrections (1st, 4th, and 5th row). Some bounds on the index of the real forms are also added to the table, for without them there are apparent contradictions. For example the indicators of the same representations of $\mathfrak{so}_{3,5}$ and $\mathfrak{so}_{5,3}$ would be different.

$\mathfrak{g}$	$\delta(\rho(\Lambda))$
$\mathfrak{su}_{k,2p-k} \quad (p \geq 2, 1 \leq k \leq p)$	$(-1)^{(p^2+k)\Lambda_p}$ $*$
$\mathfrak{u}_l^*(\mathbb{H})\ (l \geq 4)$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2\lfloor l/2\rfloor-1}}$
$\mathfrak{sl}_p(\mathbb{H})\ (p \geq 3)$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2p-1}}$
$\mathfrak{so}_{2k-1,2(l-k)+1} \quad (l \geq 2, 1 \leq k \leq \lfloor l/2\rfloor)$	$(-1)^{(k+l(l-1)/2)(\Lambda_{l-1}+\Lambda_l)}$ $*$
$\mathfrak{so}_{2k,2(l-k)+1}\ (l \geq 3, 2 \leq k \leq l)$	$(-1)^{(k+l(l+1)/2)\Lambda_l}$ $*$
$\mathfrak{so}_{2k,2(2p-k)} \quad (p \geq 2, 2 \leq k \leq p)$	$(-1)^{(k+p)(\Lambda_{2p-1}+\Lambda_{2p})}$
$\mathfrak{sp}_{k,l-k}\ (l \geq 2, 1 \leq k \leq \lfloor l/2\rfloor)$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2\lfloor(l+1)/2\rfloor-1}}$
EVI	$(-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$

For detailed computation of this table, see the author's thesis.

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