ABSTRACT

Title of dissertation: The Real-Quaternionic Indicator of Irreducible Self-Conjugate Representations of Real Reductive Algebraic Groups
and
A Comment on the Local Langlands Correspondence of $GL(2, F)$

Ran Cui, Doctor of Philosophy, 2016

Dissertation directed by: Professor Jeffrey Adams
Department of Mathematics

The real-quaternionic indicator, also called the δ indicator, indicates if a self-conjugate representation is of real or quaternionic type. It is closely related to the Frobenius-Schur indicator, which we call the ε indicator. The Frobenius-Schur indicator $\varepsilon(\pi)$ is known to be given by a particular value of the central character. We would like a similar result for the δ indicator. When G is compact, $\delta(\pi)$ and $\varepsilon(\pi)$ coincide. In general, they are not necessarily the same. In this thesis, we will give a relation between the two indicators when G is a real reductive algebraic group. This relation also leads to a formula for $\delta(\pi)$ in terms of the central character.

For the second part, we consider the construction of the local Langlands correspondence of $GL(2, F)$ when F is a non-Archimedean local field with odd residual characteristics. By re-examining the construction, we provide new proofs to some important properties of the correspondence. Namely, the construction is independent of the choice of additive character in the theta correspondence.

The Real-Quaternionic Indicator of Irreducible Self-Conjugate Representations of Real Reductive Algebraic Groups
and
A Comment on Local Langlands Correspondence of $GL(2, F)$

by

Ran Cui

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2016

Advisory Committee:
Professor Jeffrey Adams, Chair/Advisor
Professor Thomas Haines
Professor Xuhua He
Professor Jonathan Rosenberg
Professor William Gasarch, Dean's Representative

Dedication

To my parents Miao Bai and Longchen Cui.

and to my husband, Richard Rast.

Acknowledgments

I would like to express my sincere gratitude to those who helped me along the way.

To my advisor Prof. Jeffrey Adams. I can't thank you enough for your generous support, encouragement and most importantly, your patience. Your office door is always open to students. No matter how small or easy our question is, you take time to work out the answer with us. You always encourage me to ask questions, make me realize a good researcher is a curious researcher. Without your help, my thesis work would have been a frustrating and overwhelming pursuit.

I would like to thank Prof. Jonathan Rosenberg, whose course on Lie Groups laid a solid foundation for my study in this area. The clarity and precision of your lectures are tremendously inspiring to me. I also thank you for the helpful discussions about similar indicators for the C^* algebras, which opens my eyes to potential future research topics.

I would also like to thank Prof. Thomas Haines, whose RIT on the local Langlands correspondence inspired the second part of this thesis. Thank you for many generous and helpful discussions.

Thanks also go to my fellow representation theory students and friends, in no particular order: Wan-Yu Tsai, Srimathy Srinivasan, Jonathan Fernandes, Rob McLean, and Jon Cohen. Your ideas, insights and questions are tremendously enlightening to me.

I want to thank my parents for the sacrifices they made in order for me to

pursue my graduate study abroad. Your unconditional love and support is the source of my courage.

Last, to my husband Richard Rast. Thank you for making me laugh and keeping me company even through the most difficult days. I couldn't have done this without you.

Table of Contents

List of Symbols and Abbreviations

$\mathfrak{g}_0$	Lie algebra of G	3
$\mathfrak{g}$	complexification of $\mathfrak{g}_0$	3
K	maximal compact subgroup of G	3
$K(\mathbb{C})$	complexification of K	3
$(\pi^\vee, V^\vee)$	dual module of (π, V)	4
ε	Frobenius-Schur indicator	5
$(\overline{\pi}, \overline{V})$	conjugate of (π, V)	7
(π^h, V^h)	Hermitian dual of (π, V)	8
δ	real-quaternionic indicator	8
$G(\mathbb{C})$	complex connected reductive algebraic group	13
σ	real form of $G(\mathbb{C})$	13
G	real reductive algebraic Lie group	14
σ_0	real form corresponding to G	14
σ_c	compact real form commuting with σ_0	14
θ	Cartan involution of G	15
Θ_π	global character of π	17
$\pi^{h,\sigma}$	the σ-Hermitian dual of π	21
$\langle , \rangle^{\sigma_0}$	σ_0-invariant Hermitian form	22
$\langle , \rangle^{\sigma_c}$	σ_c-invariant Hermitian form	22
χ_π	central character of π	27
$z_{\rho^\vee}$	central element corresponding to $\rho^\vee$	27
$\rho^\vee$	half sum of positive co-roots	27
$(B, H, \{X_\alpha\})$	a splitting datum or pinning	32
H_f	fundamental Cartan of G	32
γ	distinguished involution in the inner class of θ	33
${}^\gamma G(\mathbb{C})$	extended group of $G(\mathbb{C})$	33
$(\widetilde{\pi}, \widetilde{V})$	induced module of π	43
Γ	Langlands parameter	48
$I(\Gamma)$	standard module attached to Γ	48
$J(\Gamma)$	unique irreducible quotient of $I(\Gamma)$	48
Λ	discrete parameter associated to Γ	49
ν	continuous parameter for Λ	49
LKT	lowest K-type	54
κ	indicator for self-duality of extended modules	59

$\mathbb{G}$	complex simple adjoint Lie group of certain types	63
$\mathbb{H}_f$	maximal torus of $\mathbb{G}$	63
$\mathbb{T}$	compact part of $\mathbb{H}_f$	63
$\mathbb{A}$	split part of $\mathbb{H}_f$	65
W_γ	Weyl group of $\mathbb{T}\gamma$ in $\mathbb{G}$	67
$\widetilde{W_\gamma}$	cover of W_γ	69
$W_\mathbb{K}$	Weyl group of $\mathbb{T}$ in $\mathbb{K}$	70
$\Delta(\mathbb{G}, \mathbb{H}_f)$	root system of $\mathbb{H}_f$ in $\mathbb{G}$	71
$\Delta^\vee(\mathbb{G}, \mathbb{H}_f)$	co-root system of $\mathbb{H}_f$ in $\mathbb{G}$	71
$R(\mathbb{G}, \mathbb{H}_f)$	root lattice of $\mathbb{H}_f$ in $\mathbb{G}$	71
$R^\vee(\mathbb{G}, \mathbb{H}_f)$	co-root lattice of $\mathbb{H}_f$ in $\mathbb{G}$	71
$\Delta_{cx}(\mathbb{G}, \mathbb{H}_f)$	set of complex roots in $\Delta(\mathbb{G}, \mathbb{H}_f)$	71
$R_\mathbb{K}$	root lattice of $\Delta(\mathbb{K}, \mathbb{T})$	71
$R_\mathbb{K}^\vee$	co-root lattice of $\Delta(\mathbb{K}, \mathbb{T})$	71
W_{aff}	affine Weyl group	73
$\rho(\Lambda)$	irreducible representation of $\mathfrak{g}$ with highest weight Λ	83
F	non-Archimedean local field with characteristic 0	92
$\mathcal{W}_F$	Weil group of F	92
$\mathfrak{G}_2(F)$	2-dimensional, semi-simple Deligne representations of the Weil group of F	92
$\boldsymbol{\mathcal{A}}_2(F)$	the set of irreducible smooth representations of $GL(2, F)$	92
$L(\pi, s)$	L-function of π	93
$\epsilon(\pi, s, \psi)$	local ϵ-factor of π	93
$\mathfrak{G}_2^1(F)$	reducible, 2-dimensional, semi-simple Deligne representations of $\mathcal{W}_F$	93
$\boldsymbol{\mathcal{A}}_2^1(F)$	irreducible non-cuspidal smooth representations of $GL(2, F)$	93
$\mathfrak{G}_2^0(F)$	the set of irreducible, 2-dimensional, semi-simple Deligne representation of $\mathcal{W}_F$	93
$\boldsymbol{\mathcal{A}}_2^0(F)$	irreducible cuspidal smooth representations of $GL(2, F)$	93
$(E/F, \chi)$	admissible pair	95
$\mathbb{P}(F)$	set of F-isomorphism classes of admissible pairs $(E/F, \chi)$	96
$\mathcal{S}(X)$	Schwartz space of X	99
$Mp(W)$	metaplectic cover of $Sp(W)$	102

Part I

The Real-Quaternionic Indicator of Irreducible Self-Conjugate Representations of Real Reductive Algebraic Groups

Chapter 1: Introduction

The notion of the Real-Quaternionic indicator was introduced by Iwahori in [2]. He used it as a tool for finding all real irreducible representations of a real Lie algebra $\mathfrak{g}_0$. The simple elegance of this indicator as well as its application to quantum physics motivated many mathematicians to study how to compute it. For instance, see [3], [1], [4], [5], etc. However, a simple formula for this indicator is still missing.

In this thesis, we establish a simple relation between the Real-Quaternionic indicator and the Frobenius-Schur indicator. For finite dimensional representations of real reductive Lie groups, this relation gives a formula for the Real-Quaternionic indicator through a well-known formula for the Frobenius-Schur indicator.

The motivational proof for the main results in this thesis will be given in Chapter 3 under the assumption that π is unitary.

The cases of finite-dimensional $(\mathfrak{g}, K)$-module and infinite-dimensional $(\mathfrak{g}, K)$-module will be treated separately. The reason for this is that the c-invariant Hermitian form behaves differently under the two assumptions. For example, the c-invariant Hermitian form exists and is positive-definite for all finite-dimensional $(\mathfrak{g}, K)$-modules but this is not true for infinite-dimensional $(\mathfrak{g}, K)$-modules.

The relation between δ-indicator and ε-indicator for finite-dimensional $(\mathfrak{g}, K)$-

modules is given in Theorem 4.4.2 and Theorem 4.5.5. The main results for infinite-dimensional $(\mathfrak{g}, K)$-modules are Theorem 5.2.5, Theorem 5.2.9, and Theorem 5.2.14.

Chapter 2: Preliminaries

Let G be a real reductive algebraic group (Definition 2.4.4). The real Lie algebra of G is denoted $\mathfrak{g}_0$ and its complexification $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$$

Fix once and for all a choice of maximal compact subgroup of G, and let it be denoted K. The complexification of K is denoted $K(\mathbb{C})$ (Definition 2.3.1).

Definition 2.0.1 (Definition 0.3.8 [6])**.** A $(\mathfrak{g}, K)$-*module* is a pair (π, V) with V a complex vector space and π a map

$$\pi : \mathfrak{g} \cup K \to \mathrm{End}(V)$$

satisfying

1. $\pi|_{\mathfrak{g}}$ is a complex linear Lie algebra representation, and $\pi|_K$ is a group representation;

2. every vector $v \in V$ is K-finite, i.e. $\dim\langle \pi(k)v \rangle < \infty$

3. the differential of the action of K is equal to the restriction to $\mathfrak{k}_0$ of the action of $\mathfrak{g}$;

4. $k \cdot (X \cdot v) = (Ad(k)X) \cdot (k \cdot v), \quad \forall k \in K, X \in \mathfrak{g}, v \in V$

A $(\mathfrak{g}, K)$-module V is called *admissible* if all K-types are of finite multiplicity.

A *morphism* of $(\mathfrak{g}, K)$-modules is a linear map intertwining both the action of K and of $\mathfrak{g}$.

Similarly, we can define $(\mathfrak{g}_0, K)$-modules

Definition 2.0.2 (Definition 2.10 [7])**.** An $(\mathfrak{g}_0, K)$-*module* is a complex vector space V that is at the same time a representation of K and of the real Lie algebra $\mathfrak{g}_0$, subject to the conditions in Definition 2.0.1 with $\mathfrak{g}$ replaced by $\mathfrak{g}_0$.

Remark 1. A $(\mathfrak{g}_0, K)$-module (π, V) is naturally a $(\mathfrak{g}, K)$-module, with the action of $\mathfrak{g}$ on V defined as:

$$(X \otimes z) \cdot v = zX \cdot v \qquad \forall X \in \mathfrak{g}_0$$

Definition 2.0.3 (Definition 8.5.1 [6])**.** If (π, V) is a $(\mathfrak{g}, K)$-module, set

$$V^\vee = \{f : V \to \mathbb{C} | \dim\langle \pi^\vee(K) f \rangle < \infty \ \}$$

here $\pi^\vee(k)$ acts by the algebraic dual representation,

$$(\pi^\vee(k)f)(v) = f(\pi(k^{-1})v) \quad \forall k \in K$$

and

$$(\pi^\vee(X)f)(v) = f(-\pi(X)v) \quad \forall X \in \mathfrak{g}$$

$(\pi^\vee, V^\vee)$ is called the *dual* module.

The module $(\pi^\vee, V^\vee)$ is indeed a $(\mathfrak{g}, K)$-module, see Lemma 8.5.2 in [6]. Moreover,

$$(\pi^{\vee\vee}, V^{\vee\vee}) \cong (\pi, V)$$

canonically.

2.1 Defining The Frobenius-Schur Indicator

Definition 2.1.1 (Frobenius-Schur indicator)**.** Let (π, V) be an irreducible self-dual $(\mathfrak{g}, K)$-module. There exists an invariant bilinear form B on V. The *Frobenius-Schur indicator* ε is defined to be:

$$\varepsilon(\pi) = \begin{cases} 1 & B \text{ is symmetric} \\ -1 & B \text{ is skew-symmetric} \end{cases}$$

We say $\varepsilon(\pi) = 0$ if π is not self-dual.

Remark 2. π self-dual implies the existence of a bilinear form B. Let $\psi : (\pi, V) \to (\pi^\vee, V^\vee)$ be any isomorphism. Define $B : V \times V \to \mathbb{C}$ to be:

$$B(v, w) := \psi(v)(w)$$

It is easy to see that B is bilinear and invariant under the actions of $\mathfrak{g}$ and K. And because π is irreducible, B is unique up to a complex multiple. We call the representations with Frobenius-Schur indicator 1 orthogonal; and those with $\varepsilon = -1$ symplectic.

We will show that B is either symmetric or skew-symmetric. An important tool is Schur's lemma for $(\mathfrak{g}, K)$-modules.

Theorem 2.1.1. *Any irreducible representation (π, V) of G is admissible (Definition 2.0.1).*

Lemma 2.1.2 (Schur's Lemma for $(\mathfrak{g}, K)$ modules)**.** *Suppose (π, V) is an irreducible complex $(\mathfrak{g}, K)$ module, and $\varphi : V \to V$ is a $(\mathfrak{g}, K)$ module isomorphism. Then there is a constant $\lambda \in \mathbb{C}$ such that $\varphi(v) = \lambda v$ for all $v \in V$.*

Proof. Let W be a δ-isotypic subspace in V, $\delta \in \hat{K}$. By assumption of irreducibility hence admissibility (Theorem 2.1.1), W is finite dimensional. Because φ is a $(\mathfrak{g}, K)$ invariant map, φ preserves W. Meaning that we can find a eigenvalue of φ on W. Let λ be that eigenvalue and w be a eigenvector of λ. Define:

$$W_\lambda = \{v \in V | \varphi(v) = \lambda v\}$$

It is clear that this space is a $(\mathfrak{g}, K)$ invariant subspace of V. The irreducibility of V implies that $W_\lambda = 0$ or $W_\lambda = V$. Because of the existence of w, W_λ is forced to be V. □

The isomorphism ψ induces $\psi^\vee : (\pi^{\vee\vee}, V^{\vee\vee}) \to (\pi^\vee, V^\vee)$, where $\psi^\vee(F) = F \circ \psi$. Let $\iota : V \to V^{\vee\vee}$ be the canonical isomorphism taking v to F_v where $F_v(f) = f(v)$. Composing $\psi^\vee$ with ι, we obtain another isomorphism $\xi(\psi) = \psi^\vee \circ \iota : (\pi, V) \to (\pi^\vee, V^\vee)$. By Schur's lemma, $\xi(\psi) = c\psi$ for some $c \in \mathbb{C}^*$. Next we show $c = \pm 1$.

A brief calculation proves that $\xi(\xi(\psi)) = \psi$:

$$\begin{aligned}[\xi(\xi(\psi))(v)](w) &= [[\xi(\psi)]^\vee \circ \iota(v)](w) = [F_v \circ \xi(\psi)](w) = [F_v \circ [\psi^\vee \circ \iota]](w) \\ &= F_v(\psi^\vee(F_w)) = F_v(F_w \circ \psi) = F_w \circ \psi(v) = \psi(v)(w)\end{aligned}$$

Since all maps involved here are linear, it's not hard to see that $\xi(c\psi) = c\xi(\psi)$. Therefore the equation above gives: $c^2\psi = \psi$, which implies $c = \pm 1$. Using this fact, we now prove $B(w,v) = \pm B(v,w)\ \forall v, w \in V$.

$$\xi(\psi)(v)(w) = [\psi^\vee \circ \iota(v)](w) = [\psi^\vee(F_v)](w) = [F_v \circ \psi](w) = \psi(w)(v) = B(w,v)$$

Therefore $B(w,v) = cB(v,w)$, i.e., $B(w,v) = \pm B(v,w)$.

2.2 Defining The Real-Quaternionic Indicator

To define the Real-Quaternionic indicator, we consider a self-conjugate $(\mathfrak{g}_0, K)$-module (π, V). First, we define the conjugate representation:

Definition 2.2.1 (Conjugate representation)**.** Let $\tau : \mathbb{C} \to \mathbb{C}$ be a complex conjugation and set:

$$\overline{V} = \mathbb{C} \otimes_{\mathbb{C},\tau} V$$

The $(\mathfrak{g}_0, K)$-module action on $\overline{V}$ is as follows:

$$\overline{\pi}(X)(z \otimes v) = z \otimes (\pi(X)v) \ \ \forall X \in \mathfrak{g}_0, v \in V, z \in \mathbb{C}$$

$$\overline{\pi}(k)(z \otimes v) = z \otimes (\pi(k)v) \ \ \forall k \in K, v \in V, z \in \mathbb{C}$$

The $(\mathfrak{g}_0, K)$-module $(\overline{\pi}, \overline{V})$ is the *conjugate representation* of (π, V).

Another way to define the conjugate representation is the dual of the Hermitian dual.

Definition 2.2.2 (Hermitian dual)**.** If (π, V) is a $(\mathfrak{g}_0, K)$-module, set

$$V^h = \{\eta : V \to \mathbb{C} | \xi(u+v) = \xi(u) + \xi(v), \xi(zv) = \overline{z}\xi(v), \dim\langle \pi^h(K)\xi \rangle < \infty\}$$

here π^h acts as follows:

$$[\pi^h(k)\xi](v) = \xi(\pi(k^{-1})v) \quad \forall k \in K$$

$$[\pi^h(X)\xi](v) = \xi(\pi(-X)v) \quad \forall X \in \mathfrak{g}_0$$

We say (π^h, V^h) is the *Hermitian Dual* of (π, V)

Remark 3. There is a version of "Hermitian dual" for $(\mathfrak{g}, K)$-modules. It's called the σ-Hermitian dual, see Definition 4.2.2 with the action of $K(\mathbb{C})$ restricted to K.

Lemma 2.2.1 (Alternative definition for conjugate representation)**.** *Let (π, V) be a $(\mathfrak{g}_0, K)$-module. We can define the* conjugate representation *of π to be $(\overline{\pi}, \overline{V}) := ([\pi^h]^\vee, [V^h]^\vee)$.*

Remark 4. The two definitions of conjugate representation are equivalent, essentially because $V^{\vee\vee} \cong V$. Notice that this may not be true for the algebraic dual V^* instead of $V^\vee$ if V is infinite-dimensional.

Lemma 2.2.2. *Any two of the three operations: dual, Hermitian dual, and conjugation, compose into the third. I.e.:*

$$(\overline{\pi}, \overline{V}) = ([\pi^h]^\vee, [V^h]^\vee) = ([\pi^\vee]^h, [V^\vee]^h)$$

$$(\pi^h, V^h) = (\overline{\pi^\vee}, \overline{V^\vee}) = ([\overline{\pi}]^\vee, [\overline{V}]^\vee)$$

$$(\pi^\vee, V^\vee) = (\overline{\pi^h}, \overline{V^h}) = ([\overline{\pi}]^h, [\overline{V}]^h)$$

Definition 2.2.3. Let (π, V) be an irreducible $(\mathfrak{g}_0, K)$-module. We say π is *self-conjugate* if $\pi \cong \overline{\pi}$.

Definition 2.2.4 (Real-Quaternionic indicator)**.** Let (π, V) be an irreducible self-conjugate $(\mathfrak{g}_0, K)$-module. There exists a non-zero $(\mathfrak{g}_0, K)$ equivariant conjugate linear map $\mathcal{J} : V \to V$. For such a map, there exists $c \in \mathbb{R}^*$ such that $\mathcal{J}^2(v) = cv$ $\forall v \in V$. The *Real-Quaternionic indicator* δ is defined to be:

$$\delta(\pi) = \text{sgn}(c)$$

We say $\delta(\pi) = 0$ if π is not self-conjugate.

We will prove that this definition is valid. The argument is similar to what we did for the Frobenius-Schur indicator.

Let $\psi : (\pi, V) \to (\overline{\pi}, \overline{V})$ be an isomorphism, $\iota : V \to \overline{V}$ be $\iota(v) = 1 \otimes v$. Consequently $\iota^{-1} : \overline{V} \to V$ is $\iota^{-1}(z \otimes v) = \overline{z}v$. Let

$$\mathcal{J} = \iota^{-1} \circ \psi : V \to V$$

It is easy to see that $\mathcal{J}$ is $(\mathfrak{g}_0, K)$ equivariant and conjugate linear. Therefore, $\mathcal{J}^2$ is $(\mathfrak{g}_0, K)$ equivariant and linear. By Schur's lemma, $\mathcal{J}^2(v) = cv$ for some $c \in \mathbb{C}^*$ and for all $v \in V$. Now we prove that $c \in \mathbb{R}^*$:

$$\mathcal{J}^3(v) = \mathcal{J}(\mathcal{J}^2(v)) = \mathcal{J}(cv) = \bar{c}\mathcal{J}(v)$$

$$= \mathcal{J}^2(\mathcal{J}(v)) = c\mathcal{J}(v)$$

Moreover, any map which satisfies the same conditions that $\mathcal{J}$ does is a complex multiple of $\mathcal{J}$, also $(z\mathcal{J})^2 = |z|\mathcal{J}^2$ for all $z \in \mathbb{C}$. Thus conclude that the definition is valid.

We are going to present an alternative definition, which is also the motivational definition for the name Real-Quaternionic indicator. We first define what it means for a $(\mathfrak{g}_0, K)$-module (π, V) to be of real/quaternionic type.

Definition 2.2.5. Let (π, V) be an irreducible self-conjugate $(\mathfrak{g}_0, K)$-module. We say (π, V) is *of real type* if there exists an irreducible $(\mathfrak{g}_0, K)$-module (π_0, W) with W a real vector space such that $\pi \cong \pi_0 \otimes_{\mathbb{R}} \mathbb{C}$; we say (π, V) is of quaternionic type if there is an irreducible $(\mathfrak{g}_0, K)$-module (ρ, U) over $\mathbb{H}$ such that $\pi = Res^{\mathbb{H}}_{\mathbb{C}}\rho$ (here Res denote the restriction of scalars).

The notation $\pi_0 \otimes_{\mathbb{R}} \mathbb{C}$ is interpreted as follows:

$$V = W \otimes_{\mathbb{R}} \mathbb{C}, \qquad \pi(x) = \pi_0(x) \otimes 1$$

Definition 2.2.6 (Alternative definition for δ-indicator)**.** Let (π, V) be an irreducible self-conjugate $(\mathfrak{g}_0, K)$-module. We define the Real-Quaternionic indicator $\delta(\pi)$ as:

$$\delta(\pi) = \begin{cases} 1 & \pi \text{ is of real type} \\ \\ -1 & \pi \text{ is of quaternionic type} \end{cases}$$

Claim 1. The above two definitions for the Real-Quaternionic indicator are equivalent.

Proof. First, we claim that π is of real type if and only if there exists a non-zero $(\mathfrak{g}_0, K)$ equivariant conjugate linear map $\mathcal{J} : V \to V$ such that $\mathcal{J}^2 = c$ for some $c \in \mathbb{R}^{>0}$.

To prove this claim, suppose π is of real type. Then there exists a real irreducible $(\mathfrak{g}_0, K)$-module (π_0, W) such that $(\pi, V) \cong (\pi_0, W \otimes_{\mathbb{R}} \mathbb{C})$. Define the map $\mathcal{J} : V \to V$ to be:

$$\mathcal{J}(w \otimes z) = w \otimes \overline{z}$$

By definition, one can quickly deduce that $\mathcal{J}$ is conjugate linear, $(\mathfrak{g}_0, K)$ equivariant and $\mathcal{J}^2 = 1$.

Conversely, suppose there exists a non-zero $(\mathfrak{g}_0, K)$ equivariant conjugate linear map $\mathcal{J} : V \to V$ such that $\mathcal{J}^2 = c$ for some $c \in \mathbb{R}^{>0}$. Take the eigenspace of $\sqrt{c}$:

$$W = \{v \in V | \mathcal{J}(v) = \sqrt{c}v\}$$

W is clearly an $\mathbb{R}$ vector space and invariant under π. We denote the restriction of π on W to be (π_0, W). This is an irreducible real $(\mathfrak{g}_0, K)$-module and $\pi \cong \pi_0 \otimes_{\mathbb{R}} \mathbb{C}$. Thus conclude the claim we made in the beginning of this proof is true.

The fact that π is of quaternionic type if and only if there exists a non-zero $(\mathfrak{g}_0, K)$ invariant conjugate linear map $\mathcal{J} : V \to V$ such that $\mathcal{J}^2 = c$ for some $c \in \mathbb{R}^{<0}$ follows from this claim and the implicit fact that if an irreducible complex $(\mathfrak{g}_0, K)$-module is self-conjugate it can either be of real type or of quaternionic type. □

2.3 Complexifications

The main tool we will be using requires the interaction between different real forms (which will be defined in Section 2.4). Therefore it is important to define the complexifications.

Definition 2.3.1 (Complexification of compact Lie groups)**.** [7, P14-15] Define $R(K, \mathbb{C})$ to be the algebra generated by matrix coefficients of finite-dimensional irreducible representations of K. Define $K(\mathbb{C})$ to be the set of all $\mathbb{C}$-algebra homomorphisms: $R(K, \mathbb{C}) \to \mathbb{C}$. We call this algebraic group the *complexification* of

K.

Definition 2.3.2. Let (π, V) be a representation of K. After picking a basis of V, we can write it as:

$$\pi : k \mapsto (\pi_{ij}(k))$$

Define the representation $(\pi(\mathbb{C}), V)$ of $K(\mathbb{C})$ as:

$$\pi(\mathbb{C}) : s \mapsto (s(\pi_{ij}))$$

It is clear that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\quad\iota\quad} & K(\mathbb{C}) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi(\mathbb{C})} & \\ & GL(n,\mathbb{C}) & \end{array}$$

Definition 2.3.3. [7, Corollary 2.18] Let (π, V) be a $(\mathfrak{g}, K)$ module. We can extend π to a $(\mathfrak{g}, K(\mathbb{C}))$-module, we call it $(\pi(\mathbb{C}), V)$, with actions of $K(\mathbb{C})$ defined as above. The compatibility conditions are the same as in Definition 2.0.1.

Theorem 2.3.1. *[7, P15-16]*

1. *The construction of complexification is a covariant functor on compact Lie groups.*

2. *Every locally finite continuous representation π of K extends uniquely to an algebraic representation $\pi(\mathbb{C})$ of $K(\mathbb{C})$ on the same space; and every algebraic representation of $K(\mathbb{C})$ restricts to a locally finite continuous representation of K.*

3. *This extension defines an equivalence of categories from* $(\mathfrak{g}_0, K)$*-modules to* $(\mathfrak{g}, K)$*-modules (Remark 1) and from* $(\mathfrak{g}, K)$*-modules to* $(\mathfrak{g}, K(\mathbb{C}))$*-modules.*

2.4 Real Forms

We can complexify a real algebraic group and obtain a complex algebraic group. On the other hand, complex Lie groups have real forms. In this section, we will define the concept of real form and some related notions. We follow [7] closely here.

Definition 2.4.1 (Complex Connected Reductive Algebraic Groups)**.** [7, Section 3] We say $G(\mathbb{C})$ is a *complex connected reductive algebraic group* if $G(\mathbb{C})$ satisfies the following:

1. it is a subgroup of the group of $n \times n$ invertible matrices, specified as the zero locus of a collection of polynomial equations in the matrix entries and the inverse of the determinant
2. it has no nontrivial normal subgroup consisting of unipotent matrices, and
3. it is connected as a Lie group

Remark 5. The second criteria is equivalent to

$2'$ After appropriate change of basis in $\mathbb{C}^n$, the group $G(\mathbb{C})$ is preserved by

$$\sigma_c(g) = (g^*)^{-1}$$

(inverse conjugate transpose) of $GL(n, \mathbb{C})$.

Definition 2.4.2 (Real Forms)**.** A *real form* of a complex Lie group $G(\mathbb{C})$ is an anti-holomorphic Lie group automorphism σ of order 2:

$$\sigma : G(\mathbb{C}) \to G(\mathbb{C}) \qquad\qquad \sigma^2 = \mathrm{Id}$$

Remark 6. The differential $d\sigma$ of σ at identity is a real form of $\mathfrak{g}$. By abuse of notation, we will say σ is a real form of $\mathfrak{g}$.

Definition 2.4.3. Given a real form σ, the *group of real points* is

$$G = G(\mathbb{R}, \sigma) = G(\mathbb{C})^{\sigma}$$

Definition 2.4.4. A *real reductive algebraic group* G (which we will call "real group" for short) is the group of real points of a complex connected reductive algebraic group.

Remark 7. Take $\sigma_c : G(\mathbb{C}) \to G(\mathbb{C})$ to be $\sigma_c(g) = (g^*)^{-1}$ after appropriate change of basis. It is a real form, so it defines a real group

$$G(\mathbb{R}, \sigma_c) = G(\mathbb{C}) \cap U(n)$$

Since $G(\mathbb{C})$ is Zariski closed in $GL(n, \mathbb{C})$ then $G(\mathbb{R})$ is closed in $U(n)$ therefore it is compact.

Definition 2.4.5. This real form is called the *compact real form* of $G(\mathbb{C})$ and it's unique up to conjugation by an inner automorphism of $G(\mathbb{C})$.

The real group G we fixed from the very beginning has a real form, we denote it as σ_0. By Definition 2.4 $G = G(\mathbb{R}, \sigma_0) = G(\mathbb{C})^{\sigma_0}$. It is well known that there

exists a compact real form σ_c such that

$$\sigma_0 \circ \sigma_c = \sigma_c \circ \sigma_0$$

this compact real form is unique up to conjugacy by $G = G(\mathbb{C})^{\sigma_0}$.

Definition 2.4.6 (Cartan involution)**.** The composition of the two real forms mentioned above

$$\theta := \sigma_0 \circ \sigma_c \tag{2.1}$$

θ is an algebraic involution of $G(\mathbb{C})$, and is called a *Cartan involution* for the real form σ_0. The group $K = G^\theta$ is a maximal compact subgroup of G and its complexification is a reductive algebraic group $K(\mathbb{C}) = G(\mathbb{C})^\theta$.

Chapter 3: Unitary Representations

Unitary representations play an important role in this paper. Many results and proofs are generalizations of those of unitary representations.

Theorem 3.0.1. *Let (π, V) be an irreducible unitary representation of G; then $\varepsilon(\pi) = \delta(\pi)$.*

Proof. First, we note that if π is Hermitian, then π is self-conjugate if and only if π is self-dual. This is a direct consequence of Lemma 2.2.2.

Now assume $\varepsilon(\pi)$ and $\delta(\pi)$ are both non-zero, we will construct a non-zero

conjugate linear equivariant map $\mathcal{J} : V \to V$ and compute the sign of $\mathcal{J}^2$. We describe this construction.

Let $B : V \times V \to \mathbb{C}$ be an invariant bilinear form on V since $\pi \cong \pi^\vee$. Let $\langle , \rangle$ be an invariant positive-definite Hermitian form on V. Define $\mathcal{J}$ by the condition:

$$B(v, w) = \langle v, \mathcal{J}(w) \rangle \tag{3.1}$$

the map $\mathcal{J} : V \to V$ is conjugate linear, G equivariant and non-zero. By the discussion after Definition 2.2.4, there exists $c \in \mathbb{R}$ such that $\mathcal{J}(v) = cv$ for all $v \in V$. We compute $\mathrm{sgn}(c) = \delta(\pi)$.

$$\langle \mathcal{J}(v), \mathcal{J}(w) \rangle = B(\mathcal{J}(v), w) = \varepsilon(\pi) B(w, \mathcal{J}(v)) = \varepsilon(\pi) \langle w, \mathcal{J}^2(v) \rangle$$

$$= \varepsilon(\pi) c \langle w, v \rangle \quad \forall v, w \in V$$

implies

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\varepsilon(\pi)c) = \mathrm{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\langle w, v \rangle}\right) = \mathrm{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\overline{\langle v, w \rangle}}\right)$$

$$\forall v, w \in V \text{ such that } \langle v, w \rangle \neq 0$$

Set $v = w$. The right hand side equal to 1 because the form $\langle , \rangle$ is positive definite. Therefore

$$\varepsilon(\pi) = \delta(\pi)$$

□

Remark 8. This proof can be done in a more general setting, the group here does not have to be a real reductive Lie group. But for convenience, we will still stay in the framework of real reductive Lie groups unless specified.

The contra-positive of Theorem 3.0.1 gives an interesting non-unitarity criterion.

Corollary 3.0.2. *Let π be an irreducible $(\mathfrak{g}_0, K)$-module; if $\varepsilon(\pi) \neq \delta(\pi)$ then π is not unitary.*

3.1 Compact Groups

Theorem 3.0.1 easily leads to the following result for compact groups:

Corollary 3.1.1. *Let G be a compact group (specifically, G can be a finite group) and let π be an irreducible representation of G, then $\varepsilon(\pi) = \delta(\pi)$.*

By analyzing the symmetric/skew-symmetric form on V (see [4] for details) we can compute $\delta(\pi)$ the following way:

Proposition 3.1.2 (Proposition 6.8, [4])**.** *Let G be a compact Lie group, (π, V) be an irreducible complex representation of G with character $\Theta_\pi \colon G \to \mathbb{C}$; then*

$$\int \Theta_\pi(g^2)\, dg = \begin{cases} 1 & \Leftrightarrow\ V \text{ is of real type} \\ 0 & \Leftrightarrow\ V \text{ is of complex type} \\ -1 & \Leftrightarrow\ V \text{ is of quaternionic type} \end{cases}$$

or simply:

$$\delta(\pi) = \int \Theta_\pi(g^2)\, dg$$

Here we use the normalized measure on G.

Apply Proposition 3.1.2 to finite groups, we get the following:

Proposition 3.1.3. *Let G be a finite group, (π, V) be an irreducible representation of G and Θ_π be the global character of (π, V). Then*

$$\varepsilon(\pi) = \delta(\pi) = \frac{1}{|G|} \sum_{g \in G} \Theta_\pi(g^2).$$

Chapter 4: Finite-Dimensional $(\mathfrak{g}_0, K)$-Modules

4.1 Introduction

It is a well known fact that the Frobenius-Schur indicator of finite-dimensional representations of connected complex reductive groups is given by a particular value of the central character. In this section, we will establish a relationship between the Frobenius-Schur indicator and the Real-Quaternionic indicator of finite-dimensional self-conjugate representations of G, and thus give a formula for the Real-Quaternionic indicator.

In section 4.3, the formula for Frobenius-Schur indicator, namely $\varepsilon(\pi) = \chi_\pi(z(\rho^\vee))$, is stated in Theorem 4.3.3. The rest of section 4 is devoted to establishing a relationship between ε and δ. It turns out

$$\delta(\pi) = \chi_\pi(x^2)\varepsilon(\pi) \tag{4.1}$$

for Hermitian representations. $x \in K$ is the strong real form given by G (Definition 4.4.1). This equation therefore gives a formula for the Real-Quaternionic

indicator of finite-dimensional Hermitian representations

$$\delta(\pi) = \chi_\pi(x^2)\chi_\pi(z(\rho^\vee))$$

The intuition behind Equation 4.1 lies in Lemma 4.3.2 and Remark 11. ε is independent of the real form but δ is dependent; therefore it is natural that the two are linked by the strong real form x. The way x appears in this relationship is also very natural. In the proof of Theorem 3.0.1, we used the positive-definite invariant Hermitian form that unitary representations possess in order to show $\varepsilon = \delta$. In general, Hermitian forms may not be positive-definite. However, as long as we stick with finite-dimensional $(\mathfrak{g}_0, K)$-modules, we will always have a c-invariant Hermitian form that is positive-definite. The extra term we have to introduce in order to use the c-form is exactly x. The strong real form x relates the ordinary invariant Hermitian form and the c-invariant Hermitian form.

The case of non-Hermitian modules only appears when the group G is not of equal rank. In that case, we introduce the extended group ${}^\gamma G$ (Definition 4.4.4). We then compute the δ-indicator of the induced module, which is the same as the δ-indicator of the original module.

4.2 Hermitian $(\mathfrak{g}_0, K)$-Modules

Definition 4.2.1. We call a $(\mathfrak{g}_0, K)$-module (π, V) *Hermitian* when it is isomorphic to its Hermitian dual (Definition 2.2.2), i.e., $\pi \cong \pi^h$. Or equivalently, there exists a Hermitian form $\langle , \rangle$ on V such that it is $(\mathfrak{g}_0, K)$-invariant.

In this section, we assume the $(\mathfrak{g}_0, K)$-module (π, V) is irreducible, Hermitian

and self-conjugate. By Lemma 2.2.2, these imply that π is self-dual. Therefore both $\varepsilon(\pi)$ and $\delta(\pi)$ exist.

Let $\langle , \rangle$ be a $(\mathfrak{g}_0, K)$ invariant Hermitian form, $B : V \times V \to \mathbb{C}$ a $(\mathfrak{g}_0, K)$ invariant bilinear form. Define $\mathcal{J} : V \to V$ as in the proof of Theorem 3.0.1:

$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

$\mathcal{J}$ is clearly conjugate linear, $(\mathfrak{g}_0, K)$ equivariant and non-zero. By the same calculation we did for Theorem 3.0.1, we obtain:

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\langle w, v \rangle}\right) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\overline{\langle v, w \rangle}}\right) \tag{4.2}$$

However, the determining of the sign function is not easy since the Hermitian form may not be positive-definite. The goal of the rest of this section is to determine the sign of the right hand side.

In order to do so, we invoke the c-invariant Hermitian form.

4.2.1 Hermitian Forms on $(\mathfrak{g}, K(\mathbb{C}))$-Modules

Our main objects of study are $(\mathfrak{g}_0, K)$-modules (π, V). However, in this section, a convenient setting are the $(\mathfrak{g}, K(\mathbb{C}))$-modules $(\pi(\mathbb{C}), V)$ which are the complexifications of (π, V). They are essentially the same thing, see Theorem 2.3.1.

The relation between δ and ε indicator depends fundamentally on the relation between two different invariant Hermitian forms on $(\pi(\mathbb{C}), V)$. Namely the σ_0 invariant Hermitian form and the σ_c invariant Hermitian form. We will define these notions.

Consider the real forms σ_0 and σ_c of $G(\mathbb{C})$ introduced at the end of section 2.4. They are naturally real structures of the pair $(\mathfrak{g}, K(\mathbb{C}))$ (for details about real structure see Section 8 of [7]). Suppose σ is a real structure of $(\mathfrak{g}, K(\mathbb{C}))$; we can define the notion of σ-Hermitian dual and σ-invariant Hermitian form.

Definition 4.2.2. Suppose (π, V) is a $(\mathfrak{g}, K(\mathbb{C}))$-module. The *$\sigma$-Hermitian dual* of (π, V) is denoted $(\pi^{h,\sigma}, V^h)$

$$\pi^{h,\sigma} : K(\mathbb{C}) \to GL(V^h), \qquad \pi^{h,\sigma}(k) = [\pi(\sigma(k^{-1})]^h \ \ \forall k \in K(\mathbb{C})$$

The operator $\pi(k)^h$ is the Hermitian transpose of the operator $\pi(k)$ (see Section 8 in [7]).

Remark 9.

1. The relation $\sigma_0 \circ \sigma_c = \theta$ implies $[\pi^{h,\sigma_0}]^{h,\sigma_c} \cong \pi^\theta$

2. Recall that π is a $(\mathfrak{g}_0, K)$-module, $\pi(\mathbb{C})$ denotes the corresponding $(\mathfrak{g}, K(\mathbb{C}))$-module (Definition 2.3.3). The following equation relates their Hermitian duals:

   $$(\pi^h)(\mathbb{C}) \cong \pi(\mathbb{C})^{h,\sigma}$$

   where σ is the real form corresponding to the real group G.

Definition 4.2.3. [7, Definition 8.6] Suppose (π, V) is a $(\mathfrak{g}, K(\mathbb{C}))$-module, a *$\sigma$-invariant Hermitian form* on V is a Hermitian pairing $\langle , \rangle^\sigma$ on V satisfying

$$\langle X \cdot v, w\rangle^\sigma = \langle v, -\sigma(X) \cdot w\rangle^\sigma \quad \forall X \in \mathfrak{g}, v, w \in V$$

$$\langle k \cdot v, w\rangle^\sigma = \langle v, \sigma(k^{-1}) \cdot w\rangle^\sigma \quad \forall k \in K(\mathbb{C}), v, w \in V$$

such a form may be identified with an intertwining operator

$$T \in \mathrm{Hom}_{\mathfrak{g},K(\mathbb{C})}(\pi, \pi^{h,\sigma})$$

with the Hermitian condition: $T = T^h$

Lemma 4.2.1. *A $(\mathfrak{g}, K(\mathbb{C}))$-module (π, V) has a σ-invariant Hermitian form if and only if $\pi \cong \pi^{h,\sigma}$.*

In this thesis, we are particularly interested in two invariant Hermitian forms, namely $\langle , \rangle^{\sigma_0}$ and $\langle , \rangle^{\sigma_c}$. We sometimes call the former the "ordinary Hermitian form" and the latter "c-invariant Hermitian form". It is the focus of this section to establish the connection between the two invariant Hermitian forms.

First we introduce the concept of extended pair.

Definition 4.2.4. [7, Definition 8.12] Let μ be an automorphism of the pair $(\mathfrak{g}, K(\mathbb{C}))$, and λ be an element of $K(\mathbb{C})$ such that

$$\mu^2 = \mathrm{Ad}(\lambda) \quad \lambda \in K(\mathbb{C})$$

is an inner automorphism of the pair $(\mathfrak{g}, K(\mathbb{C}))$. The corresponding *extended group* is the extension

$$1 \to K(\mathbb{C}) \to {}^{\mu}K(\mathbb{C}) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

with a specified generator μ_1 mapping to $1 \in \mathbb{Z}/2\mathbb{Z}$, and subject to the relation $\mu_1 k \mu_1^{-1} = \mu(k)$ for $k \in K(\mathbb{C})$, and $\mu_1^2 = \lambda$. This extended group acts by automorphisms on $\mathfrak{g}$, and we call $(\mathfrak{g}, {}^{\mu}K(\mathbb{C})$ an *extended pair*.

Proposition 4.2.2. *Suppose (π, V) is an finite-dimensional $(\mathfrak{g}, K(\mathbb{C}))$-module, and V has a non-degenerate σ_c-invariant Hermitian form $\langle , \rangle^{\sigma_c}$, unique up to a real multiple. Then:*

1. *The following are equivalent:*

   (a) *There is a non-degenerate σ_0-invariant Hermitian form $\langle , \rangle^{\sigma_0}$. It is unique up to a real multiple.*

   (b) *(π, V) is isomorphic to its twist by θ:*

   $$D : (\pi, V) \cong (\pi^\theta, V)$$

   In this case D is unique up to a complex multiple.

   Assuming $\theta^2 = Ad(\lambda)$ for some $\lambda \in K(\mathbb{C})$, then (a) and (b) are also equivalent to the following.

   (c) *The $(\mathfrak{g}, K(\mathbb{C}))$-module V extends to a $(\mathfrak{g}, {}^\theta K(\mathbb{C}))$-module with θ acting as D subject to the additional requirement that $D^2 = \pi(\lambda)$. If any such extensions exist then there are exactly two. The operators D differ by multiplication by ± 1.*

Assuming from now on the condition of (c) is satisfied, and the operator D is chosen to be that in (c). Furthermore, assume $\langle \lambda \cdot v, \lambda \cdot w \rangle^{\sigma_c} = \langle v, w \rangle^{\sigma_c}$.

2 *There is a non-zero complex number ξ so that*

$$D^{-1} = \xi D$$

on V, and $|\xi|^2 = 1$

3 The form $\langle v, w\rangle' = \langle D(v), D(w)\rangle^{\sigma_c}$ *is again a* σ_c*-invariant Hermitian form, so there exists a real number* ω *such that*

$$\langle D(v), D(w)\rangle^{\sigma_c} = \omega\langle v, w\rangle^{\sigma_c}$$

4 The scalar ω *is* ± 1*.*

5 If ζ *is a square root of* $\omega\overline{\xi}$ *Then*

$$\langle v, w\rangle^{\sigma_0} := \zeta^{-1}\langle D(v), w\rangle^{\sigma_c} = \zeta\langle v, D(w)\rangle^{\sigma_c}$$

is a σ_0*-invariant Hermitian form on* V*.*

Proof. It is easy to verify that any σ_c-invariant Hermitian form differ by a real multiple from $\langle , \rangle^{\sigma_c}$, because the condition $\langle v, w\rangle^{\sigma_c} = \overline{\langle v, w\rangle^{\sigma_c}}$.

$(a) \Rightarrow (b)$: The existence of σ-Hermitian form is equivalent to the isomorphism between π and π^{h,σ_0}. From the assumption of this proposition and 1, we know:

$$\pi \cong \pi^{h,\sigma_0} \qquad \pi \cong \pi^{h,\sigma_c}$$

Therefore

$$\pi^{h,\sigma_0} \cong \pi^{h,\sigma_c}$$

Take the σ_c-Hermitian dual of both sides:

$$[\pi^{h,\sigma_0}]^{h,\sigma_c} \cong [\pi^{h,\sigma_c}]^{h,\sigma_c} \Rightarrow \pi^{\theta} \cong \pi$$

$(b) \Rightarrow (a)$: Following similar arguments as above, this is not hard to show.

$(b) \Leftrightarrow (c)$: The reason for assuming $\theta^2 = \mathrm{Ad}(\lambda)$ is because it's required for defining the extended pair. The equivalence is an immediate consequence of basic Clifford theory.

Proof of 2: We know that θ is the Cartan involution, more specifically, it is an involution. So $\theta^2 = \mathrm{Id}$ on $K(\mathbb{C})$ and $\lambda \in Z(K(\mathbb{C}))$. This implies that $\pi(\lambda) = D^2$ acts on V by a complex scalar, call it ξ^{-1}. Then it is immediate that $D^{-1} = \xi D$.

It remains to show that $|\xi| = 1$. Consider the equation $D^{-1} = \xi D$, square both sides: $D^{-2} = \xi^2 D^2$. For any $v, w \in V$ we have

$$\langle D^{-2}(v), w\rangle^{\sigma_c} = \langle \xi^2 D^2(v), w\rangle^{\sigma_c} \Rightarrow \langle \lambda^{-1} \cdot v, w\rangle^{\sigma_c} = \langle \xi^2 \lambda \cdot v, w\rangle^{\sigma_c}$$

$$\Rightarrow \langle v, \lambda \cdot w\rangle^{\sigma_c} = \xi^2 \langle v, \lambda^{-1} \cdot w\rangle^{\sigma_c} \Rightarrow D^2 = \overline{\xi^2} D^{-2} \Rightarrow |\xi|^4 = 1 \Rightarrow \xi\overline{\xi} = 1$$

Proof of 3: It's clear that $\langle , \rangle'$ is sesquilinear and Hermitian. It remains to show that it is σ_c-invariant:

$$\langle X \cdot v, w\rangle' = \langle v, -\sigma_c(X) \cdot w\rangle'$$

It is useful to write out the intertwining property of the isomorphism D first:

$$D(\pi(X)v) = [\pi^\theta(X)](D(v))$$

The σ_c-invariant claim follows after a short calculation:

$$\begin{aligned}
\langle X \cdot v, w\rangle' &= \langle D(X \cdot v), D(w)\rangle^{\sigma_c} = \langle X \cdot_\theta D(v), D(w)\rangle^{\sigma_c} \\
&= \langle \theta(X) \cdot D(v), D(w)\rangle^{\sigma_c} = \langle D(v), -\sigma_c(\theta(X)) \cdot D(w)\rangle^{\sigma_c} \\
&= \langle D(v), -\theta(\sigma_c(X)) \cdot D(w)\rangle^{\sigma_c} = \langle D(v), -D(\sigma_c(X) \cdot w)\rangle^{\sigma_c} \\
&= \langle v, -\sigma_c(X) \cdot w\rangle' \quad \forall X \in \mathfrak{g}
\end{aligned}$$

A similar argument proves the σ_c-invariance for all $k \in K(\mathbb{C})$.

Since $\langle , \rangle'$ is another σ_c-invariant Hermitian for on the irreducible representation (π, V), it differ by a real scalar from $\langle , \rangle^{\sigma_c}$. We denote this real scalar ω.

Proof of 4:

$$\langle D^2(v), D^2(w)\rangle^{\sigma_c} = \omega\langle D(v), D(w)\rangle^{\sigma_c} = \omega^2\langle v, w\rangle^{\sigma_c}$$

we know that D^2 acts on V by $\pi(\lambda)$ where $\lambda \in K(\mathbb{C})$, therefore we have $\langle D^2(v), D^2(w)\rangle^{\sigma_c} = \pi(\lambda)\overline{\pi(\lambda)}\langle v, w\rangle^{\sigma_c} = \xi^{-1}\overline{\xi^{-1}}\langle v, w\rangle = \langle v, w\rangle^{\sigma_c}$. This implies $\omega^2 = 1$, i.e., $\omega = \pm 1$.

Proof of 5: The form $\langle , \rangle^{\sigma_0}$ is clearly sesquilinear, it is also Hermitian:

$$\begin{aligned}\langle w, v\rangle^{\sigma_0} &= \zeta^{-1}\langle D(w), v\rangle^{\sigma_c} = \zeta^{-1}\overline{\langle v, D(w)\rangle^{\sigma_c}} = \zeta^{-1}\overline{\omega^{-1}\langle D(v), D^2(w)\rangle^{\sigma_c}} \\ &= \zeta^{-1}\omega\overline{\langle D(v), \xi^{-1}w\rangle^{\sigma_c}} = \zeta^{-1}\omega\xi^{-1}\overline{\langle D(v), w\rangle^{\sigma_c}} = \zeta^{-1}\omega\overline{\xi}\overline{\langle D(v), w\rangle^{\sigma_c}} \\ &= \zeta\overline{\langle D(v), w\rangle^{\sigma_c}} = \overline{\zeta^{-1}\langle D(v), w\rangle^{\sigma_c}} = \overline{\langle v, w\rangle^{\sigma_0}}\end{aligned}$$

Here we used the fact that $|\zeta|^2 = 1$. It is an easy consequence of $|\xi|^2 = 1$ and $|\omega|^2 = 1$.

It remains to show that this form is σ_0-invariant.

$$\langle X \cdot v, w\rangle^{\sigma_0} = \langle v, -\sigma_0(X) \cdot w\rangle^{\sigma_0}$$

$$\begin{aligned}\langle X \cdot v, w\rangle^{\sigma_0} &= \zeta^{-1}\langle D(X \cdot v), w\rangle^{\sigma_c} = \zeta^{-1}\langle \theta(X) \cdot D(v), w\rangle^{\sigma_c} \\ &= \zeta^{-1}\langle D(v), -\sigma_c(\theta(X)) \cdot w\rangle^{\sigma_c} = \zeta^{-1}\langle D(v), -\sigma_0(X) \cdot w\rangle^{\sigma_c} \\ &= \langle v, -\sigma_0(X) \cdot w\rangle^{\sigma_0}\end{aligned}$$

Similar calculation applies to all $k \in K(\mathbb{C})$.

□

Lemma 4.2.3. *A $(\mathfrak{g}_0, K)$-module (π, V) is Hermitian if and only if the corresponding $(\mathfrak{g}, K(\mathbb{C}))$-module $(\pi(\mathbb{C}), V)$ is σ_0-Hermitian.*

Proof. It is clear that the form is σ_0-invariant for the $\mathfrak{g}$ action on V since $\mathfrak{g} = \{X + iY | X, Y \in \mathfrak{g}_0\}$ and $\sigma_0(X + iY) = X - iY$. If $\langle , \rangle$ is a $\mathfrak{g}_0$-invariant Hermitian form then it is also σ_0-invariant under the action of $\mathfrak{g}$.

The $K(\mathbb{C})$-invariance follows easily from Theorem 2.3.1 (2) and (3). The equivalence of category (Theorem 2.3.1(3)) together with the uniqueness of restriction and complexification implies $(\pi, V) \cong (\pi^h, V^h) \Rightarrow (\pi(\mathbb{C}), V) \cong (\pi(\mathbb{C})^{h,\sigma_0}, V^h)$. The lemma follows. □

4.3 The Frobenius-Schur Indicator

Theorem 4.3.1. *[8, Ch.IX, 7.2, Proposition 1] Let $G(\mathbb{C})$ be a connected complex reductive Lie group, π an irreducible finite-dimensional self-dual representation of $G(\mathbb{C})$. Then*

$$\varepsilon(\pi) = \chi_\pi(z(\rho^\vee)).$$

Here χ_π is the central character of π, $z_{\rho^\vee} = \exp(2\pi i \rho^\vee)$ and $\rho^\vee$ is the half sum of the positive co-roots.

For proof, see [9, Lemma 5.2].

Lemma 4.3.2. *Suppose G is a real form of $G(\mathbb{C})$, π is an irreducible self-dual finite-dimensional $(\mathfrak{g}, K)$-module. Let $\pi(\mathbb{C})$ be the corresponding $(\mathfrak{g}, K(\mathbb{C}))$-module,*

therefore a representation of $G(\mathbb{C})$. *Then*

$$\varepsilon(\pi) = \varepsilon(\pi(\mathbb{C})) = \chi_\pi(z_{\rho^\vee})$$

Proof. That π is self-dual if and only if $\pi(\mathbb{C})$ is self-dual is implied by the functoriality of the complexification.

Suppose π is self-dual and $\varepsilon(\pi) = 1$. Let B be a symmetric $(\mathfrak{g}, K)$ invariant bilinear form on V; then it is also invariant under the action of $\mathfrak{g}$. If we can show $\pi(\mathbb{C})$ restricted on K is also orthogonal then we can conclude that $\pi(\mathbb{C})$ is an orthogonal $(\mathfrak{g}, K(\mathbb{C}))$-module.

Since K is compact, by Theorem 3.1.1 $\delta(\pi) = \varepsilon(\pi) = 1$. So the representation is both orthogonal and of real type. Therefore $\pi : K \to O(n)$. By Theorem 2.3.1, we know that the complexification is a covariant functor, therefore $\pi(\mathbb{C}) : K(\mathbb{C}) \to O(n, \mathbb{C})$. Therefore $\pi(\mathbb{C})$ is orthogonal, i.e., $\varepsilon(\pi(\mathbb{C})) = 1$.

Since $\pi(\mathbb{C})$ can only be orthogonal or symplectic if it's self-dual, therefore it directly follows that $\varepsilon(\pi) = -1$ implies $\varepsilon(\pi(\mathbb{C})) = -1$. □

Lemma 4.3.2 implies that the formula for ε of complex groups also applies to any real form of that complex group.

Theorem 4.3.3. *Suppose* G *is a real reductive group, and* π *is an irreducible finite-dimensional self-dual representation of* G. *Then*

$$\varepsilon(\pi) = \chi_\pi(z(\rho^\vee)).$$

Remark 10. Note that $z(\rho^\vee)$ is fixed by every automorphism of $G(\mathbb{C})$, so $z(\rho^\vee) \in Z(G)$ for every real form G of $G(\mathbb{C})$.

Remark 11. The Real-Quaternionic indicator is dependent on the real form. This can be illustrated by an example. Let $G_1 = SL(2,\mathbb{R})$ and $G_2 = SU(2)$, take (π_1, V) and (π_2, V) to be the irreducible 2-dimensional representations of G_1 and G_2 respectively whose complexification is the irreducible 2-dimensional representation of $SL(2,\mathbb{C})$. The representation π_1 is clearly of real type, because the explicit action shows it preserves the 2-dimensional real vector space defined by restricting scalars of the complex vector space V. On the other hand, it is not hard to see that π_2 preserves a skew-symmetric bilinear form, since $G_2 \cong Sp(1)$ and π_2 is just the defining representation of $Sp(1) \subset \mathbb{H}^\times$ on $\mathbb{H}$. Therefore $\delta(\pi_1) = 1$ and $\delta(\pi_2) = -1$.

4.4 The Real-Quaternionic Indicator

We assume, for this section, that (π, V) is a Hermitian $(\mathfrak{g}_0, K)$-module, with $(\mathfrak{g}_0, K)$-invariant Hermitian form $\langle , \rangle$. Furthermore we assume that π has real infinitesimal character. The reason for this is to ensure the existence of the σ_c-invariant Hermitian form; see the following lemma.

Lemma 4.4.1. *For every finite-dimensional $(\mathfrak{g}, K(\mathbb{C}))$-module with real infinitesimal character ([6, Definition 5.4.11]), there always exists a σ_c-invariant Hermitian form. In particular, when G is semi-simple, or when $Z(G)$ contains no split torus, every finite-dimensional representation has real infinitesimal character.*

See Proposition 5.2.1.

Theorem 4.4.2. *Let (π, V) be an irreducible finite-dimensional $(\mathfrak{g}_0, K)$-module with*

real infinitesimal character. Assume π is both self-conjugate and Hermitian; then

$$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)$$

where χ_π is the central character of π and x is a strong real form given by the real group G.

In order to prove this theorem, we need to consider two cases: 1. when G is equal rank (Definition 4.4.1); 2. when G is unequal rank. The reason for this dichotomy is that the D and λ we choose have different properties in the two cases. Therefore the implementation of Proposition 4.2.2 in the proof of Theorem 4.4.2 is different for the two cases.

4.4.0.1 G is Equal Rank

In this section, we assume G is an equal rank group.

Definition 4.4.1. Suppose G is a real reductive algebraic group and $K = G^\theta$ is a maximal compact subgroup, then G is said to be *equal rank* if the rank of G is equal to the rank of K. Equivalently, the automorphism θ of G is inner. In this case, a *strong involution* for G is an element $x \in G$ such that $\mathrm{Ad}(x) = \theta$. The definition of *unequal rank* groups is the natural negation of the above conditions.

Thus $K = \mathrm{Cent}_G(x)$, so $x \in Z(K)$, and $x^2 \in Z(K)$. It also follows that $x^2 = z \in Z(G) \cap K$.

Proposition 4.4.3. *Let (π, V) be a finite-dimensional irreducible $(\mathfrak{g}_0, K)$-module which is both self-conjugate and Hermitian, and G an equal rank reductive algebraic*

group. Then V possesses a σ_c-invariant Hermitian form $\langle , \rangle^{\sigma_c}$, and we can define form $\langle , \rangle^{\sigma_0}$ such that

$$\langle v, w \rangle^{\sigma_0} = \zeta^{-1} \langle x \cdot v, w \rangle^{\sigma_c}$$

where ζ is a square root of $\chi_\pi(z)$, and x is a strong involution for G. This form is σ_0-invariant. In particular, $\langle , \rangle^{\sigma_0}$ is an ordinary invariant Hermitian form under the actions of $(\mathfrak{g}_0, K)$.

Proof. We have already established that there exists a σ_c-invariant Hermitian form for (π, V) under these assumptions. The rest of the proposition is simply a special case of Proposition 4.2.2, taking D to be $\pi(x)$ and $\lambda = x^2 = z$. In this case $\omega = 1$, $\xi = \zeta^{-2}$ and the square root of $\omega\overline{\xi} = \xi^{-1}$ is ζ.

We are downplaying the distinction between $(\mathfrak{g}_0, K)$-modules and $(\mathfrak{g}, K(\mathbb{C}))$-modules, because the difference is not of essential importance to us and it may be a potential distraction. □

Proof of Theorem 4.4.2 when G is equal rank. Fix once and for all a σ_c-invariant Hermitian form on V and an ordinary invariant Hermitian form $\langle , \rangle^{\sigma_0}$ defined in Proposition 4.4.3.

Recall Equation 4.2 is written in terms of the ordinary invariant Hermitian form:

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle^{\sigma_0}}{\overline{\langle v, w \rangle^{\sigma_0}}} \right)$$

We rewrite this equation in terms of the σ_c-invariant Hermitian form $\langle , \rangle^{\sigma_c}$

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle x \cdot \mathcal{J}(v), \mathcal{J}(w) \rangle^{\sigma_c}}{\overline{\zeta^{-1}\langle x \cdot v, w \rangle^{\sigma_c}}} \right) = \operatorname{sgn}(\zeta^{-2}) \operatorname{sgn}\left(\frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w) \rangle^{\sigma_c}}{\overline{\langle x \cdot v, w \rangle^{\sigma_c}}} \right)$$

The definition of $\mathcal{J}$ implies directly that is is a $(\mathfrak{g}_0, K)$ invariant map. Since the strong involution x is an element of $Z(K)$, we have $x \cdot \mathcal{J}(v) = \mathcal{J}(x \cdot v)$.

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle \mathcal{J}(x \cdot v), \mathcal{J}(w)\rangle^{\sigma_c}}{\overline{\langle x \cdot v, w\rangle^{\sigma_c}}}\right)$$

Set $w = x \cdot v$, then

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^{\sigma_c}}{\overline{\langle w, w\rangle^{\sigma_c}}}\right)$$

The c-invariant Hermitian is positive-definite, therefore

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-2}) = \chi_\pi(x^2)$$

Here we used the fact that the central character of a self-dual representation is ± 1 valued. □

4.4.0.2 G is Unequal Rank

For an unequal rank group G, the Cartan involution θ is not inner. We can still talk about strong involutions, and use them to define $\langle , \rangle^{\sigma_0}$ from $\langle , \rangle^{\sigma_c}$. However the strong real form of G won't be an element of G, it is an element of the extended group ${}^{\gamma}G$ of G. Here γ is a certain distinguished involution in the inner class of θ. We will define the distinguished involution and the extended group.

Definition 4.4.2. A *splitting datum* or *pinning* is a set $(B, H, \{X_\alpha\})$ where B is a Borel subgroup, H is a Cartan subgroup contained in B and $\{X_\alpha\}$ is a set of root vectors for the simple roots of H in B.

Definition 4.4.3. An involution of G is said to be *distinguished* if it preserves a pinning.

We now define the distinguished involution γ we need. Here we follow [7] closely.

Choose a maximal torus $T_f \subset K$ and let $H_f = \mathrm{Cent}_G(T_f)$. This is a *fundamental Cartan* of G. Let $R = R(G, H_f)$ be the roots of H_f in G. We can choose a system of positive roots R^+ so that $\theta(R^+) = R^+$. This defines a Borel B_f containing H_f.

Remark 12. The choice of R^+ here can result in a different pinning hence a different distinguished involution. However, the two distinguished involutions will be conjugate to each other by an element of $G(\mathbb{C})$. Suppose $\mathcal{P}$ and $\mathcal{P}'$ are two different pinnings corresponding to the different choices of R^+, then there exists a $g \in G(\mathbb{C})$ such that conjugating by g takes $\mathcal{P}$ to $\mathcal{P}'$. The resulting distinguished involution γ and γ' are conjugate by $\mathrm{Ad}(g)$. Therefore γ and γ' are equivalent as real forms and $\gamma^2 = (\gamma')^2$.

The simple root vectors $\Pi = \{X_\alpha\}$ are chosen in such a way that

$$\gamma(X_\alpha) = X_{\theta\alpha} \quad \alpha \in \Pi$$

Let γ be the distinguished involution in the inner class of θ that preserves the pinning (B_f, H_f, Π). We see that

$$\gamma^2 = 1, \quad \gamma \circ \theta = \theta \circ \gamma, \quad \gamma \circ \sigma_c = \sigma_c \circ \gamma. \tag{4.3}$$

With this distinguished involution, we define the extended group of $G(\mathbb{C})$.

Definition 4.4.4. The *extended group* ${}^\gamma G(\mathbb{C})$ for $G(\mathbb{C})$ is the semi-direct product

$${}^\gamma G(\mathbb{C}) = G(\mathbb{C}) \rtimes \{1, \gamma\}$$

According to (4.3), γ preserves G, K, $K(\mathbb{C})$. We can therefore define all the corresponding extended groups ${}^{\gamma}G$, ${}^{\gamma}K$, and ${}^{\gamma}K(\mathbb{C})$.

The *strong real form for the real form* G is an element

$$x = x_0\gamma \in G(\mathbb{C})\gamma = {}^{\gamma}G(\mathbb{C})\backslash G(\mathbb{C})$$

with the property that

$$\mathrm{Ad}(x)|_{G(\mathbb{C})} = \theta.$$

Proposition 4.4.4. *The strong real form x for the real form G satisfies:*

1. $x_0 \in K$ *and* $x_0^2 = x^2 \in Z(K)$.

2. $x \in {}^{\gamma}K$.

We would like to define $\langle , \rangle^{\sigma_0}$ from $\langle , \rangle^{\sigma_c}$ the same way we did in the last section. However, the expression $x \cdot v$ does not make sense now that x is in the extended group. To make sense of this action, we need to extend the $(\mathfrak{g}_0, K)$-module (π, V) to a $(\mathfrak{g}_0, {}^{\gamma}K)$-module. By Clifford theory, (π, V) can be extended to a $(\mathfrak{g}_0, {}^{\gamma}K)$-module if and only if $\pi \cong \pi^{\gamma}$. Lemma 4.4.1 implies that $\pi \cong \pi^{h,\sigma_c}$; we are under the assumption that all representations are Hermitian $\pi \cong \pi^h := \pi^{h,\sigma_0}$. Therefore $\pi \cong [\pi^{h,\sigma_0}]^{h,\sigma_c} \cong \pi^{\theta} \cong \pi^{\gamma}$, the first isomorphism is because $\sigma_0 \circ \sigma_c = \theta$, the second is because θ is inner to γ. The $(\mathfrak{g}_0, K)$-module (π, V) can have two different extensions, we call them π_1 and π_2. The difference between them is $\pi_1(\gamma) = -\pi_2(\gamma)$.

Before giving the proof of Theorem 4.4.2, we need to prove some properties of the extended modules.

Lemma 4.4.5. *Let (π, V) be an irreducible finite-dimensional self-dual representation of G and $\pi \cong \pi^\gamma$. Then it has two extensions, denoted π_1 and π_2. Furthermore, π_1 and π_2 are self-dual.*

Proof. The distinction between finite-dimensional representations and their Harish-Chandra modules is trivial for our purpose, so upon proving the claim for representations, the same claim for $(\mathfrak{g}_0, K)$-modules should be immediate.

First of all we claim that $(\pi_1)^\vee \cong \pi_1$ or π_2.

Proof of Claim: by definition of dual representation,

$$(\pi_1)^\vee(\gamma) \cdot f(v) = f(\pi_1(\gamma^{-1}) \cdot v)$$

This action of γ extends $\pi^\vee$ because it intertwines $\pi^\vee$ and $(\pi^\vee)^\gamma$:

$$\begin{array}{ccc} V^\vee & \xrightarrow{(\pi_1)^\vee(\gamma)} & V^\vee \\ {\scriptstyle \pi^\vee}\downarrow & & \downarrow{\scriptstyle (\pi^\vee)^\gamma} \\ V^\vee & \xrightarrow{(\pi_1)^\vee(\gamma)} & V^\vee \end{array}$$

It is elementary to check this:

$$\begin{aligned} [(\pi^\vee)^\gamma(g) \cdot [(\pi_1)^\vee(\gamma) \cdot f]](v) &= [\pi^\vee(\gamma(g)) \cdot [(\pi_1)^\vee(\gamma) \cdot f]](v) \\ &= (\pi_1)^\vee(\gamma) \cdot f(\pi_1(\gamma(g^{-1})) \cdot v) \\ &= f(\pi_1(\gamma^{-1} \cdot \gamma \cdot g^{-1} \cdot \gamma^{-1}) \cdot v) = f(\pi_1(\gamma g^{-1}) \cdot v) \end{aligned}$$

$$\begin{aligned} [(\pi_1)^\vee(\gamma) \cdot [\pi^\vee(g) \cdot f]](v) &= [\pi^\vee(g) \cdot f](\pi_1(\gamma^{-1}) \cdot v) = f(\pi(g^{-1})\pi_1(\gamma^{-1}) \cdot v) \\ &= f(\pi_1(\gamma^{-1}g^{-1}) \cdot v) \end{aligned}$$

So $(\pi_1)^\vee(\gamma)$ extends $\pi^\vee$, and since $\pi^\vee \cong \pi$, $(\pi_1)^\vee(\gamma)$ also extends π. It is either π_1 or π_2.

It turns out $(\pi_1)^\vee \cong \pi_1$, we will now prove this.

Since π is self-dual, it is equivalent to prove: $(\pi_1)^\vee \cong (\pi^\vee)_1$. The notation $(\pi^\vee)_1$ is chosen so that $\pi_1 \cong (\pi^\vee)_1$:

$$(\pi^\vee)_1(g) \cdot f = \psi[\pi_1(g) \cdot \psi^{-1}(f)]$$

where ψ is the representation isomorphism $\psi : \pi \cong \pi^\vee$.

The idea of the proof is using the highest weight vector of $\pi^\vee$ and analyzing how $(\pi_1)^\vee(\gamma)$ and $(\pi^\vee)_1(\gamma)$ act on it. The highest weight vector is defined with respect to H_f and the pinning (H_f, B_f, Π) (see discussion after Definition 4.4.3).

The reason for using the highest weight vector is that $(\pi^\vee)_1(\gamma)$ acts on it by a scalar

$$(\pi^\vee)_1(\gamma) \cdot \varphi = c \cdot \varphi;$$

see Lemma 4.4.6 below for proof. Consequently, $(\pi^\vee)_2(\gamma) \cdot \varphi = -c \cdot \varphi$.

It is convenient to use proof by contradiction, i.e. to prove $(\pi_1)^\vee \ncong (\pi^\vee)_2$. Note that a necessary condition for $(\pi_1)^\vee \cong (\pi^\vee)_2$ is:

$$(\pi_1)^\vee(\gamma) \cdot \varphi = -c \cdot \varphi$$

because for any representation isomorphism $\varphi : (\pi_1)^\vee \cong (\pi^\vee)_2$, φ sends highest weights to highest weights and therefore preserves the scalar by which γ acts. To prove $(\pi_1)^\vee \ncong (\pi^\vee)_2$, we just need to show the action of $(\pi_1)^\vee$ and that of $(\pi^\vee)_2$ on the highest weight vector are different.

Let v be the highest weight vector of π with highest weight λ; we claim that the dual basis $f_{\sigma_{\omega_0} \cdot v}$ in $V^\vee$ is the highest weight vector for $\pi^\vee$ with highest weight

$-\omega_0 \cdot \lambda = \lambda$, where σ_{ω_0} is the element in the Tits group with respect to the pinning (H_f, B_f, Π) (see discussion after Definition 4.4.3). The reason we use this element is:

$$\gamma(\sigma_{\omega_0}) = \sigma_{\omega_0}$$

We first verify that $f_{\sigma_{\omega_0}\cdot v}$ is indeed the highest weight vector for $\pi^\vee$, for $H \in \mathfrak{h}(\mathbb{C})$:

$$\begin{aligned}
\pi^\vee(H) \cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v) &= f_{\sigma_{\omega_0}\cdot v}(-H \cdot \sigma_{\omega_0} \cdot v) \text{ (by Definition of } \pi^\vee) \\
&= f_{\sigma_{\omega_0}\cdot v}(-\sigma_{\omega_0}\sigma_{\omega_0}^{-1} H \sigma_{\omega_0} \cdot v) \text{ (by } (\mathfrak{g}(\mathbb{C}), K(\mathbb{C}) \text{ module)} \\
&= f_{\sigma_{\omega_0}\cdot v}(-\sigma_{\omega_0} \cdot \omega_0^{-1}(H) \cdot v) \text{ (by Definition)} \\
&= -\lambda(\omega_0(H)) f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v) \\
&= -\omega_0(\lambda)(H) f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v). \text{ (Weyl group acts on weights)}
\end{aligned}$$

Because the representation π is self-dual, $-\omega_0(\lambda) = \lambda$, this implies:

$$\pi^\vee(H) \cdot f_{\sigma_{\omega_0}\cdot v} = \lambda(H) f_{\sigma_{\omega_0}\cdot v}.$$

Let's see how $(\pi_1)^\vee(\gamma)$ acts on $f_{\sigma_{\omega_0}\cdot v}$:

$$\begin{aligned}
(\pi_1)^\vee(\gamma) \cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v) &= f_{\sigma_{\omega_0}\cdot v}(\pi_1(\gamma^{-1}) \cdot \sigma_{\omega_0} \cdot v) \\
&= f_{\sigma_{\omega_0}\cdot v}(\gamma(\sigma_{\omega_0}) \cdot \pi_1(\gamma) \cdot v) \\
&= f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot c \cdot v) = c \cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v).
\end{aligned}$$

However, on the other hand:

$$(\pi^\vee)_2(\gamma) \cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v) = -[\psi(\pi_1(\gamma) \cdot \psi^{-1}(f_{\sigma_{\omega_0}\cdot v}))](\sigma_{\omega_0} \cdot v).$$

Since $\psi^{-1}(f_{\sigma_{\omega_0}\cdot v})$ is going to be a highest weight vector of π and $\pi_1(\gamma)$ acts on that by a scalar c, we have:

$$(\pi^\vee)_2(\gamma) \cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v) = -c \cdot f_{\sigma_{\omega_0}\cdot v}(\sigma_{\omega_0} \cdot v).$$

Therefore, we have a contradiction:

$$(\pi_1)^\vee(\gamma) \cdot f_{\sigma_{\omega_0} \cdot v} \neq (\pi^\vee)_2(\gamma) \cdot f_{\sigma_{\omega_0} \cdot v}.$$

it has to be:

$$(\pi_1)^\vee \cong (\pi^\vee)_1 \cong \pi_1.$$

□

Lemma 4.4.6. *Let v be a highest weight vector of π with highest weight λ,*

$$\pi_1(\gamma) \cdot v = c \cdot v, \quad c \in \mathbb{C}^*.$$

Proof. Let ϕ denote this isomorphism $\pi \cong \pi^\gamma$

$$\phi : V \to V$$

$$\pi^\gamma(g) \cdot \phi(v) = \phi(\pi(g) \cdot v).$$

Set the action of γ to be $\pi_1(\gamma)(v) := \phi(v)$.

Consider the weight space decomposition of π. Let $\{\lambda, \lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the weights and $\{V, V_1, V_2, \cdots, V_n\}$ be the corresponding weight spaces, with λ the highest weight, V the highest weight space. The map γ sends V to the weight space of λ^γ:

$$\pi(H) \cdot \gamma(v) = \gamma(\gamma(H) \cdot v) = \gamma(\lambda^\gamma(H) \cdot v) = \lambda^\gamma(H) \cdot \gamma(v)$$

the notation $\lambda^\gamma(H)$ is defined as $\lambda(\gamma(H))$ By the uniqueness of the weight space decomposition and the fact that γ is an isomorphism, we know that the action of γ on V permutes the weights:

$$\{\lambda^\gamma, \lambda_1^\gamma, \cdots \lambda_n^\gamma\} = \{\lambda, \lambda_1, \cdots, \lambda_n\}.$$

It turns out that twisting by γ does not change the order of the weights, thanks to the considerate choice of based root datum and γ in [7]. More importantly, λ^γ is still the highest weight of π^γ. We give a brief proof of this fact: Since λ is the highest weight, we have:

$$\lambda - \lambda_i = \sum_{j \in S} \alpha_j$$

where $\alpha_j \in R^+$ and R^+ is the set of positive roots chosen to be preserved by θ. S is some non-empty set, meaning we are summing over some subset of R^+.

Twisting by γ, we have:

$$\lambda^\gamma - \lambda_i^\gamma = \sum_{j \in S} \alpha_j^\gamma$$

By the construction of γ from θ, we conclude γ also preserves R^+ and thus α_j^γ are again positive roots, so

$$\lambda^\gamma \succ \lambda_i^\gamma, \quad \forall i$$

Therefore $\lambda^\gamma = \lambda$ and

$$\gamma(v) = \pi_1(\gamma) \cdot v = c \cdot v, \quad c \in \mathbb{C}^*$$

□

Lemma 4.4.7. *Let (π, V) be an irreducible finite-dimensional c-Hermitian representation of G satisfying $(\pi, V) \cong (\pi^\gamma, V)$. Then it has two extensions π_1 and π_2 and they are c-Hermitian.*

Proof. The extended module π_1 being c-Hermitian is equivalent to

$$\pi_1 \cong [\pi_1]^{h, \sigma_c}$$

It is easy to see that $[\pi_1]^{h,\sigma_c}$ is also an extension of π. Because there are only two extensions of π, we will prove the above isomorphism by contradiction. We will show

$$[\pi_1]^{h,\sigma_c} \ncong \pi_2$$

Suppose there exists such an isomorphism

$$\psi : \pi_2 \to [\pi_1]^{h,\sigma_c}$$

$$[\pi_1]^{h,\sigma_c}(\gamma) \cdot \psi(v) = \psi(\pi_2(\gamma)v) \tag{4.4}$$

Let $v \in J$ such that $\pi_1(\gamma)v = cv$ some $c \in \mathbb{C}^*$. Such vector exists because γ acts as an isomorphism $D : \pi \to \pi^\gamma$ and V is finite-dimensional.

By definition, $\pi_2(\gamma)v = -cv$. We also know that $\psi(v)(v) \neq 0$ because the c-Hermitian form on J is positive definite.

Equation 4.4 left hand side evaluate on v is

$$[\pi_1]^{h,\sigma_c}(\gamma) \cdot \psi(v)(v) = \psi(v)(\pi_1(\gamma^{-1})v) = \psi(v)(c^{-1}v) = \overline{c^{-1}}\psi(v)(v)$$

right hand side evaluate on v equals to

$$-c\psi(v)(v)$$

We know that $|c| = 1$, therefore in order for there to be an isomorphism ψ, $\overline{c^{-1}} = c$ has to equal to $-c$ which implies $c = 0$. Contradiction. □

It is clear that $\pi_1 \cong \pi_1^\gamma$, so an immediate consequence is that $\pi_1 \cong [\pi_1^{h,\sigma_c}]^\gamma \cong \pi_1^{h,\sigma_0}$. From the previous Lemmas, we see that the extended representations inherit many properties of the original representation. This is crucial in the proof of Theorem 4.4.2. First, we state the parallel of Proposition 4.4.3.

Proposition 4.4.8. *Let (π, V) be a finite-dimensional irreducible $(\mathfrak{g}_0, K)$-module which is both self-conjugate and Hermitian, G an unequal rank reductive algebraic group. Suppose V possesses a σ_c-invariant Hermitian form $\langle , \rangle^{\sigma_c}$, then we define form $\langle , \rangle^{\sigma_0}$ such that*

$$\langle v, w\rangle^{\sigma_0} = \zeta^{-1}\langle x \cdot v, w\rangle^{\sigma_c}$$

where ζ is a square root of $\chi_\pi(x^2)$, and $x \in {}^\gamma G$ is a strong involution for G. This form is σ_0-invariant. In particular, $\langle , \rangle^{\sigma_0}$ is an ordinary invariant Hermitian form under the actions of $(\mathfrak{g}_0, K)$.

Proof of Theorem 4.4.2 when G is unequal rank. Fix a σ_c-invariant Hermitian form for (π, V), denote it $\langle , \rangle^{\sigma_c}$. Define $\langle , \rangle^{\sigma_0}$ to be as in Proposition 4.4.8. Through the same steps as the proof when G is equal rank, we get

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle x \cdot \mathcal{J}(v), \mathcal{J}(w)\rangle^{\sigma_c}}{\overline{\langle x \cdot v, w\rangle^{\sigma_c}}}\right)$$

Recall that one important step in the proof of the theorem when G is equal rank is that $\mathcal{J}$ is equivariant under the action of x and therefore we can take the $x\cdot$ inside and set $w = x \cdot v$ thus proving the theorem. This is still true even though x is in the extended group ${}^\gamma G$. All the Lemmas we introduced before are for the proof of the fact that $\mathcal{J}$ is equivariant under the action of x.

Recall the definition of $\mathcal{J}$ is

$$B(v, w) = \langle v, \mathcal{J}(w)\rangle^{\sigma_0}$$

Lemma 4.4.5 implies that B is also invariant under the action of γ hence of x. The short discussion after Lemma 4.4.7 implies that the Hermitian form $\langle , \rangle^{\sigma_0}$ is also

invariant under the action of γ therefore of x. With these in mind, we see that the map $\mathcal{J}$ intertwines the action of x:

$$\langle v, x \cdot \mathcal{J}(w)\rangle^{\sigma_0} = \langle x^{-1} \cdot v, \mathcal{J}(w)\rangle^{\sigma_0} = B(x^{-1} \cdot v, w) = B(v, x \cdot w) = \langle v, \mathcal{J}(x \cdot w)\rangle^{\sigma_0}$$

$\Rightarrow x \cdot \mathcal{J}(v) = \mathcal{J}(x \cdot v)$. Therefore

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle \mathcal{J}(x \cdot v), \mathcal{J}(w)\rangle^{\sigma_c}}{\overline{\langle x \cdot v, w\rangle^{\sigma_c}}}\right)$$

Let $w = x \cdot v$,

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^{\sigma_c}}{\overline{\langle w, w\rangle^{\sigma_c}}}\right) = \mathrm{sgn}(\zeta^{-2}) = \chi_\pi(x^{-2}) = \chi_\pi(x^2)$$

This completes the proof of Theorem 4.4.2. □

4.5 non-Hermitian Representations

We depart from the friendly territory of Hermitian representations, and now consider (π, V) to be a finite-dimensional irreducible non-Hermitian representation. We still assume $\pi \cong \overline{\pi}$.

Having no invariant Hermitian form is a problem for us, since the techniques we have used rely on Hermitian forms. Our approach for this section is to construct a representation $\widetilde{\pi}$ from π which is Hermitian. We then find the δ-indicator for $\widetilde{\pi}$, which turns out to equal to the δ-indicator of the original representation.

One implicit property of the group G in this section is that it is unequal rank. This is because all finite-dimensional irreducible representations of an equal rank group are Hermitian. Recall for an unequal rank group G, we defined a non-trivial distinguished involution γ and extended groups ${}^{\gamma}G(\mathbb{C})$, ${}^{\gamma}G$, ${}^{\gamma}K$, and ${}^{\gamma}K(\mathbb{C})$.

The $(\mathfrak{g}_0, K)$-module (π, V) cannot be extended to a $(\mathfrak{g}_0, {}^\gamma K)$-module, however by Clifford theory, it can be induced irreducibly to a $(\mathfrak{g}_0, {}^\gamma K)$-module.

Definition 4.5.1. Let $(\widetilde{\pi}, \widetilde{V})$ be the induced module $\widetilde{\pi} = \mathrm{Ind}_{(\mathfrak{g}_0, K)}^{(\mathfrak{g}_0, {}^\gamma K)} \pi$. We will often simplify the notation and write $\widetilde{\pi} = \mathrm{Ind}(\pi)$.

Lemma 4.5.1. *The induced $(\mathfrak{g}_0, {}^\gamma K)$-module $(\widetilde{\pi}, \widetilde{V})$ is irreducible, Hermitian and self-conjugate.*

Proof. Lemma 4.4.1 says that $\pi \cong \pi^{h,\sigma_c}$, therefore $\pi^{h,\sigma_0} \cong \pi^\gamma$. The assumption that π is non-Hermitian also implies that $\pi \ncong \pi^\gamma$. Therefore Clifford theory implies that $\mathrm{Ind}\pi$ is irreducible.

It is elementary to verify that $\widetilde{\pi^h} \cong \widetilde{\pi}^h$. We know that $\widetilde{\pi^h} \cong \widetilde{\pi^\gamma} \cong \widetilde{\pi}$. Therefore $\widetilde{\pi} \cong \widetilde{\pi}^h$, i.e., $\widetilde{\pi}$ is Hermitian.

For the proof of self-conjugacy, we want to show that $\overline{\widetilde{\pi}} \cong \widetilde{\overline{\pi}}$. We view $\overline{\widetilde{V}}$ as

$$\mathbb{C} \otimes_{\mathbb{C},\tau} \widetilde{V} = \mathbb{C} \otimes_{\mathbb{C},\tau} (V \oplus \gamma V) = (\mathbb{C} \otimes_{\mathbb{C},\tau} V) \oplus (\mathbb{C} \otimes_{\mathbb{C},\tau} \gamma V)$$

and $\widetilde{\overline{V}}$ as

$$\overline{V} \oplus \gamma \overline{V} = (\mathbb{C} \otimes_{\mathbb{C},\tau} V) \oplus \gamma(\mathbb{C} \otimes_{\mathbb{C},\tau} V)$$

Define map

$$f : \overline{\widetilde{V}} \to \widetilde{\overline{V}}$$

such that

$$(z \otimes v_1) \oplus (w \otimes \gamma v_2) \mapsto (z \otimes v_1) \oplus \gamma(w \otimes v_2)$$

It is elementary to check this map intertwines the two representations.

Given the assumption that $\pi \cong \overline{\pi}$, we have

$$\widetilde{\overline{\pi}} \cong \widetilde{\overline{\pi}} \cong \widetilde{\pi}$$

i.e., $\widetilde{\pi}$ is self-conjugate. □

This module $(\widetilde{\pi}, \widetilde{V})$ turns out to have better properties than (π, V). So we have reason to believe that the same proof technique we used for Hermitian modules is applicable here too.

Proposition 4.5.2. *Let $D : \widetilde{V} \to \widetilde{V}^{\theta}$ be an isomorphism of the two $(\mathfrak{g}_0, {}^{\gamma}K)$-modules, satisfying the property that $D^2(v) = \lambda \cdot v$, where $\lambda \in {}^{\gamma}K$. Let $\langle , \rangle^{\sigma_c}$ be a non-degenerate positive definite σ_c invariant Hermitian form on V. Define*

$$\langle \tilde{v}, \tilde{w} \rangle^{\sigma_0} := \zeta^{-1} \langle D(\tilde{v}), \tilde{w} \rangle^{\sigma_c} = \zeta \langle \tilde{v}, D(\tilde{w}) \rangle^{\sigma_c}$$

with ζ, ω and ξ are defined the same as in Proposition 4.2.2. The form $\langle , \rangle^{\sigma_0}$ is a σ_0-invariant Hermitian form on $\widetilde{V}$.

The proof of Proposition 4.2.2 can be transferred here completely with no essential change.

Theorem 4.5.3. *Suppose $(\widetilde{\pi}, \widetilde{V})$ is an irreducible Hermitian self-conjugate $(\mathfrak{g}_0, {}^{\gamma}K)$-module. Then the δ-indicator of $(\widetilde{\pi}, \widetilde{V})$ is given by*

$$\delta(\widetilde{\pi}) = \varepsilon(\widetilde{\pi}) \chi_{\widetilde{\pi}}(x^2).$$

Proof. Since $\widetilde{\pi}$ is both Hermitian and self-conjugate, it is then self-dual. Let $B : V \times V \to \mathbb{C}$ be a $(\mathfrak{g}_0, {}^{\gamma}K)$-invariant bilinear form on $\widetilde{V}$.

Fix once and for all a c-invariant Hermitian form $\langle , \rangle^{\sigma_c}$, let $\langle , \rangle^{\sigma_0}$ be defined by $\langle , \rangle^{\sigma_c}$ the way in Proposition 4.5.2.

Define $\mathcal{J} : \widetilde{V} \to \widetilde{V}$ such that

$$B(\widetilde{v}, \widetilde{w}) = \langle \widetilde{v}, \mathcal{J}(\widetilde{w}) \rangle^{\sigma_0}$$

$\mathcal{J}$ is clearly $(\mathfrak{g}_0, {}^{\gamma}K)$-invariant, conjugate linear and bijective.

A short calculation and replacing $\langle , \rangle^{\sigma_0}$ by $\langle , \rangle^{\sigma_c}$ gives:

$$\varepsilon(\widetilde{\pi})\delta(\widetilde{\pi}) = \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle x \cdot \mathcal{J}(\widetilde{v}), \mathcal{J}(\widetilde{w}) \rangle^{\sigma_c}}{\overline{\langle x \cdot \widetilde{v}, \widetilde{w} \rangle^{\sigma_c}}}\right)$$

Proposition 4.4.4 says $x \in {}^{\gamma}K$, then by definition, $\mathcal{J}$ intertwines the action of x:

$$\varepsilon(\widetilde{\pi})\delta(\widetilde{\pi}) = \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle \mathcal{J}(x \cdot \widetilde{v}), \mathcal{J}(\widetilde{w}) \rangle^{\sigma_c}}{\overline{\langle x \cdot \widetilde{v}, \widetilde{w} \rangle^{\sigma_c}}}\right).$$

Let $\widetilde{w} = x \cdot \widetilde{v}$; we have

$$\varepsilon(\widetilde{\pi})\delta(\widetilde{\pi}) = \chi_{\widetilde{\pi}}(x^{-2}) \Rightarrow \delta(\widetilde{\pi}) = \varepsilon(\widetilde{\pi})\chi_{\widetilde{\pi}}(x^2).$$

□

Lemma 4.5.4. *Suppose (π, V) is an irreducible $(\mathfrak{g}_0, K)$-module $(\widetilde{\pi}, \widetilde{V})$ its irreducibly induced $(\mathfrak{g}_0, {}^{\gamma}K)$-module. If $\delta(\pi)$ and $\delta(\widetilde{\pi})$ both exist, then $\delta(\pi) = \delta(\widetilde{\pi})$.*

Proof. Since π is self-conjugate, there exists $\mathcal{J} : V \to V$ conjugate linear and G invariant. By definition of induced representation $\widetilde{V} = V \oplus \gamma V$. Define $\widetilde{\mathcal{J}} : \widetilde{V} \to \widetilde{V}$ such that:

$$\widetilde{\mathcal{J}}(v + \gamma w) = \mathcal{J}(v) + \gamma\mathcal{J}(w), \quad \forall v \in V, \gamma w \in \gamma V$$

It is easy to see that $\widetilde{\mathcal{J}}$ is conjugate linear and ${}^{\gamma}G$-invariant. We will demonstrate the calculation for ${}^{\gamma}G$-invariance. For $g \in G$:

$$\begin{aligned}\widetilde{\mathcal{J}}(g \cdot (v + \gamma w)) &= \widetilde{\mathcal{J}}(g \cdot v + \gamma\gamma(g) \cdot w) = \mathcal{J}(g \cdot v) + \gamma\mathcal{J}(\gamma(g) \cdot w) \\ &= g \cdot \mathcal{J}(v) + \gamma\gamma(g) \cdot \mathcal{J}(w) = g \cdot \mathcal{J}(v) + g \cdot \gamma\mathcal{J}(w) \\ &= g \cdot \widetilde{\mathcal{J}}(v + \gamma w)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathcal{J}}(\gamma \cdot (v + \gamma w)) &= \widetilde{\mathcal{J}}(\gamma v + zw) = \gamma\mathcal{J}(v) + z\mathcal{J}(w) = \gamma(\mathcal{J}(v) + \gamma\mathcal{J}(w)) \\ &= \gamma\widetilde{\mathcal{J}}(v + \gamma w)\end{aligned}$$

Then for $v \in V$:

$$\delta(\pi)v = \mathcal{J}^2(v) = \widetilde{\mathcal{J}}^2(v) = \delta(\widetilde{\pi})v$$

□

Theorem 4.5.5. *Suppose (π, V) is an irreducible self-conjugate $(\mathfrak{g}_0, K)$-module with real infinitesimal character and π is not Hermitian. Then the δ-indicator of π is:*

$$\delta(\pi) = \chi_\pi(x^2)\varepsilon(\widetilde{\pi})$$

where $\widetilde{\pi}$ is the irreducible module $Ind(\pi)$.

Remark 13. The indicator $\varepsilon(\widetilde{\pi})$ is understood when G is simple; in that case the Chevalley involution C is either trivial or inner to γ. The formula for $\varepsilon(\widetilde{\pi})$ is given in [9]. The author predicts that in the case of G semi-simple and reductive, a double extended group of G is needed for understanding the formula $\delta(\pi) = \chi_\pi(x^2)\varepsilon(\widetilde{\pi})$.

Chapter 5: Infinite-Dimensional $(\mathfrak{g}_0, K)$-Modules

In this chapter, our main objects of interest are the irreducible admissible representations and their Langlands parameters. In Section 5.1, we give a brief introduction to the Langlands classification for representations of G. Many notations will also be introduced here. We will then restrict our attention on the self-conjugate irreducible modules. Furthermore, we will only consider the modules with real infinitesimal character. The reason for this assumption is that the c-invariant Hermitian form is guaranteed to exist in this case, and it will be positive-definite on the lowest K-types [6, 5.4.18].

5.1 A Brief Introduction to the Langlands Classification

References for the details of the Langlands classification are [10] and [7]. Here we follow [7] closely.

Definition 5.1.1. [7, Theorem 6.1] Suppose G is the group of real points of a connected complex reductive algebraic group. Then there is a one-to-one correspondence between the infinitesimal equivalent classes of irreducible quasi-simple representations of G and G-conjugacy classes of triples

$$\Gamma = (H, \gamma, R^+_{i\mathbb{R}})$$

subject to the following requirements.

1. The group H is a Cartan subgroup of G: the group of real points of a maximal torus of $G(\mathbb{C})$ defined over $\mathbb{R}$.

2. The character γ is level one character of the ρ_{abs} double cover of H. Write $d\gamma \in \mathfrak{h}^*$ for its differential.

3. The roots $R^+_{i\mathbb{R}}$ are a positive system for the imaginary roots of H in $\mathfrak{g}$.

4. The weight $d\gamma$ is weakly dominant for $R^+_{i\mathbb{R}}$.

5. If $\alpha^\vee$ is real and $\langle d\gamma, \alpha^\vee \rangle = 0$ then $\gamma_{\mathfrak{q}}(m_\alpha) = +1$ [7, Definition 5.7].

6. If β is simple for $R^+_{i\mathbb{R}}$ and $\langle d\gamma, \beta^\vee \rangle = 0$ then β is non-compact.

We call Γ a *Langlands parameter*.

Attached to each equivalence class of Langlands parameter Γ is a standard $(\mathfrak{g}_0, K)$-module $I(\Gamma)$, and it has a unique irreducible quotient module $J(\Gamma)$. The correspondence is

$$\Gamma \leftrightarrow J(\Gamma)$$

I will give a summary of Langlands' construction of $I(\Gamma)$. First we define some notations. Let $T = H^\theta$ be the compact subgroup of H. Write $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ for the decomposition of the real Lie algebra of H into $+1$ and -1 eigenspaces of θ. Define $A = \exp(\mathfrak{a}_0)$ to be the identity component of the maximal split torus of H. Note that the group A is isomorphic to its Lie algebra and

$$H = T \times A$$

Let $MA = \mathrm{Cent}_G(A)$ be the Langlands decomposition of the centralizer of A in G. The compact group T is a compact Cartan subgroup of M, and the parameters

$$\Lambda = (T, \gamma|_{\widetilde{T}}, R^+_{i\mathbb{R}})$$

are Harish-Chandra parameters for a limit of discrete series representation $D(\Lambda) \in \widehat{M}$. Here $\widetilde{T}$ is the $\rho_{i\mathbb{R}}$ double cover of T. For more information about this cover, see [7]. Now let

$$\nu = \gamma|_A \in \widehat{A}.$$

Choose a parabolic subgroup $P = MAN$ of G such that ν is weakly dominant for the weights of $\mathfrak{a}$ in $\mathfrak{n}$. Then the standard representation $I(\Gamma)$ can be realized as

$$I_{quo}(\Gamma) = \mathrm{Ind}_P^G D(\Lambda) \otimes \nu \otimes 1$$

We also use $I_{quo}(\Gamma)$ to denote the Harish-Chandra module of the standard representation. This module has a unique irreducible quotient $J(\Gamma)$. The correspondence in Theorem 5.1.1 is $\Gamma \leftrightarrow J(\Gamma)$.

The construction of the standard module inspires another expression of the Langlands parameter. Given Langlands parameter Γ, let

$$\Lambda = (T, \gamma|_{\widetilde{T}}, R^+_{i\mathbb{R}})$$

be the discrete Langlands parameter associated to Γ, and

$$\nu = \gamma|_A \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}_0, \mathbb{C}) = \mathfrak{a}_0^*$$

be the continuous parameter for Λ. Then Γ can also be written as (Λ, ν); thus $I(\Gamma) = I(\Lambda, \nu)$ and $J(\Gamma) = J(\Lambda, \nu)$. We will be using ($\Lambda$,$\nu$) rather than Γ for the rest of this thesis.

Definition 5.1.2. We say a character ν of A is *real* if it is a real-valued character, i.e.,

$$\nu \in \mathfrak{a}_0^*(\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{R})$$

Proposition 5.1.1. *Suppose $\Gamma = (\Lambda, \nu)$ is a Langlands parameter, then*

1. *Two Langlands parameters are called equivalent if they are conjugate by G.*
2. *The lowest K-types of $I(\Lambda, \nu)$ all have multiplicity one, and they all appear in the Langlands quotient $J(\Lambda, \nu)$.*
3. *The infinitesimal character of $J(\Lambda, \nu)$ is real [6, Definition 5.4.11] if and only if $\nu \in \mathfrak{a}_0^*$ is real (Definition 5.1.2).*

5.2 Non-Unitary Modules

The benefit of the unitarity condition is that with a positive-definite invariant Hermitian form, the sign of a certain fraction is easily determined as can be seen in the proof of Theorem 3.0.1. For non-unitary modules, our tool for determining signs is the c-invariant Hermitian form. By assuming real infinitesimal character, we ensured the existence of the c-invariant form (Proposition 5.2.1).

Definition 5.2.1. Suppose $\Gamma = (\Lambda, \nu)$ is a Langlands parameter (Definition 5.1.1). The *c-Hermitian dual of* (Λ, ν) is

$$\Gamma^{h,\sigma_c} = (\Lambda, \nu)^{h,\sigma_c} = (\Lambda, \overline{\nu})$$

In particular, if the continuous parameter ν is real, then $\Gamma^{h,\sigma_c} = \Gamma$.

Proposition 5.2.1. *[7, Proposition 10.7] Suppose $\Gamma = (\Lambda, \nu)$ is a Langlands parameter for G, and $\Gamma^{h,\sigma_c} = (\Lambda, \overline{\nu})$ is the c-Hermitian dual parameter, then*

1. *The c-Hermitian dual of a standard module with a Langlands quotient is a standard module with a Langlands submodule*

$$[I(\Gamma)_{quo}]^{h,\sigma_c} = I_{sub}(\Gamma^{h,\sigma_c})$$

   and vice versa.

2. *The c-Hermitian dual of a irreducible quotient module is*

$$[J(\Gamma)]^{h,\sigma_c} = J(\Gamma^{h,\sigma_c})$$

3. *The irreducible quotient module $J(\Gamma)$ admits a c-invariant Hermitian form if and only if Γ is equivalent to Γ^{h,σ_c}.*

4. *$J(\Lambda, \nu)$ admits a c-invariant Hermitian form if and only if there exists $w \in W(G, H)$, where H is a real Cartan subgroup of G, such that $w \cdot \Lambda = \Lambda$ and $w \cdot \nu = \overline{\nu}$. In particular, if ν is real then $J(\Lambda, \nu)$ has a c-form.*

5. *Suppose ν is real, then any c-invariant Hermitian form on $J(\Lambda, \nu)$ has the same sign on the lowest K-types. In particular, the form can be chosen to be positive-definite on every lowest K-type.*

Since we assume ν is real, there always will be a c-invariant Hermitian form that is positive-definite on the lowest K-types. Therefore we can use this tool as we have used it in the finite-dimensional case. The methods of computing the δ-indicator is similar here with some additional discussion about lowest K-types.

5.2.1 Modules for Equal Rank Groups

Let G be a equal rank (Definition 4.4.1) real reductive algebraic group (Definition 2.4.4), $(\pi, J(\Lambda, \nu))$ be an irreducible admissible quotient module of G with Langlands parameter (Λ, ν) and ν real. Further assume that $J(\Lambda, \nu)$ is self-conjugate (Definition 2.2.1).

Based on our assumption, G being equal rank and ν being real, we don't have to consider the non-Hermitian modules. Because

Proposition 5.2.2. *If G is equal rank, and ν is real, then $J(\Lambda, \nu)$ is Hermitian.*

Proof. We know

$$[J(\Lambda, \nu)^{h,\sigma_c}]^{\gamma} \cong J(\Lambda, \nu)^{h,\sigma_0}$$

G being equal rank implies that γ is the identity map. Therefore

$$J(\Lambda, \nu)^{h,\sigma_c} \cong J(\Lambda, \nu)^{h,\sigma_0}.$$

By Proposition 5.2.1, ν real implies

$$J(\Lambda, \nu) \cong J(\Lambda, \overline{\nu}).$$

Therefore

$$J(\Lambda, \nu) \cong J(\Lambda, \nu)^{h,\sigma_c} \cong J(\Lambda, \nu)^{h,\sigma_0}$$

i.e., $J(\Lambda, \nu)$ is Hermitian. □

We only consider the Hermitian modules.

Recall that the δ-indicator is given by the sign of a conjugate linear invariant map $\mathcal{J}$. Let $\langle , \rangle$ be an invariant Hermitian form on $J(\Lambda, \nu)$. The assumptions that

$J(\Lambda, \nu)$ is both Hermitian and self-conjugate implies that it is also self-dual, hence there exists an invariant bilinear form B on $J(\Lambda, \nu)$. We define a conjugate linear invariant map $\mathcal{J}$ as before (Equation 3.1):

$$B(v, w) = \langle v, \mathcal{J}(w) \rangle$$

consequently we have

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\overline{\langle v, w \rangle}}\right) \tag{5.1}$$

In this case, we have invariant Hermitian forms and c-invariant Hermitian forms at our disposal. Proposition 4.2.2 still holds in the infinite-dimensional case.

Proposition 5.2.3. *Suppose $J(\Lambda, \nu)$ is an admissible $(\mathfrak{g}_0, K)$-module, and it admits a c-invariant Hermitian form $\langle , \rangle^{\sigma_c}$. If G is equal rank, then define $\langle , \rangle^{\sigma_0}$*

$$\langle v, w \rangle^{\sigma_0} = \zeta^{-1} \langle x \cdot v, w \rangle^{\sigma_c} = \zeta \langle v, x \cdot w \rangle^{\sigma_c}$$

where $x \in K$ is the strong real form given by G. This form is a σ_0-invariant Hermitian form.

This is the same Proposition as Proposition 4.4.3 without the assumption of finite-dimensionality.

Let's rewrite Equation 5.1 in terms of a c-invariant Hermitian form using the relation indicated by Proposition 5.2.3:

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\langle \mathcal{J}(v), \mathcal{J}(w) \rangle}{\overline{\langle v, w \rangle}}\right) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle x \cdot \mathcal{J}(v), \mathcal{J}(w) \rangle^c}{\overline{\zeta^{-1}\langle x \cdot v, w \rangle^c}}\right)$$

Since G is equal rank, $x \in G$. In particular, $x \in K$. By definition of $\mathcal{J}$, it intertwines the action of $(\mathfrak{g}_0, K)$. Therefore $x \cdot \mathcal{J}(v) = \mathcal{J}(x \cdot v)$ and the equation becomes

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle \mathcal{J}(x \cdot v), \mathcal{J}(w)\rangle^c}{\overline{\zeta^{-1}\langle x \cdot v, w\rangle^c}}\right)$$

Set $x \cdot v = w$

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\zeta^{-1}\langle w, w\rangle^c}}\right) = \operatorname{sgn}(\zeta^{-1}\overline{\zeta})\operatorname{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w, w\rangle^c}}\right)$$

The main goal now is to determine the sign of the fraction $\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w, w\rangle^c}}$.

Unlike the finite-dimensional case, the c-form for $J(\Lambda, \nu)$ may not be positive-definite on the entire representation. However, by Proposition 5.2.1(5) it can be made positive-definite on all of the lowest K-types [6, Definition 5.4.18]. We will call the set of lowest-K types LKT for short

Lemma 5.2.4. *The map $\mathcal{J}$ sends LKT to LKT. In particular, $\mathcal{J}$ takes λ LKT to $\overline{\lambda}$ LKT.*

Proof. Let λ be the highest weight of a lowest K-type, w be an element in the λ weight space. Denote $T \subset K$ the Cartan in K,

$$t \cdot w = \lambda(t)w \quad \forall t \in T$$

The element $\mathcal{J}(w)$ is in the $\overline{\lambda}$ weight space:

$$t \cdot \mathcal{J}(w) = \mathcal{J}(t \cdot w) = \mathcal{J}(\lambda(t)w) = \overline{\lambda(t)}\mathcal{J}(w)$$

Since λ is a character of a compact group, namely compact Cartan T of K, λ is purely imaginary valued:

$$\overline{\lambda} = -\lambda$$

therefore

$$||\lambda|| = ||-\lambda||$$

is minimal among the K-types. □

Proposition 5.2.1(5) and Lemma 5.2.4 indicates that the sign of $\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w,w\rangle^c}}$ is 1.

Theorem 5.2.5. *Let G be an equal rank real reductive algebraic group, and $J(\Lambda,\nu)$ an irreducible admissible Langlands quotient $(\mathfrak{g}_0, K)$-module with Langlands parameter $\Gamma = (\Lambda,\nu)$ satisfying the conditions in Definition 5.1.1. Further assume that ν is real and $J(\Lambda,\nu)$ is Hermitian and self-conjugate. Then*

$$\varepsilon(\pi)\delta(\pi) = \chi_\pi(x^2)$$

where $x \in G$ is the strong real form corresponding to G.

Proof. From the discussion after Proposition 5.2.1 and Lemma 5.2.4, we know:

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-1}\overline{\zeta})$$

By the definition of ζ, and the proof of Proposition 4.2.2 (5) we have

$$\varepsilon(\pi)\delta(\pi) = \mathrm{sgn}(\zeta^{-2}) = \chi_\pi(x^2)$$

Here we used the fact that the central character of a self-dual representation is ± 1 valued. □

5.2.2 Modules for Unequal Rank Groups

Let G be an unequal rank real reductive algebraic group, $J(\Lambda,\nu)$ be an irreducible admissible Langlands quotient module of G. Assume that the continuous

parameter ν is real to ensure the existence of c-form and the positive-definite property of the form on lowest K-types. We consider self-conjugate modules only.

Lemma 5.2.6. *Assume G and $J(\Lambda,\nu)$ satisfies all conditions stated in the beginning of this section. Then $J(\Lambda,\nu)$ is Hermitian if and only if $J(\Lambda,\nu)^\gamma \cong J(\Lambda,\nu)$.*

Proof. The assumption ν real implies that $J(\Lambda,\nu)$ admits an c-invariant Hermitian form by Proposition 5.2.1. This means

$$J(\Lambda,\nu) = J(\Lambda,\nu)^{h,\sigma_c}$$

Since

$$[J(\Lambda,\nu)^{h,\sigma_c}]^{h,\sigma_0} \cong J(\Lambda,\nu)^\theta \cong J(\Lambda,\nu)^\gamma$$

we have

$$J(\Lambda,\nu)^{h,\sigma_0} \cong J(\Lambda,\nu)^\gamma$$

The claim follows. □

We will divide the discussion into two parts. The first is for the Hermitian modules, second is for the non-Hermitian modules.

5.2.2.1 Hermitian Modules

Now suppose $J(\Lambda,\nu)$ is Hermitian. This together with the assumption that $J(\Lambda,\nu)$ is self-conjugate implies it is self-dual. Let $\langle,\rangle^{\sigma_0}$ be an ordinary invariant Hermitian form on $J(\Lambda,\nu)$ and B be an invariant bilinear form. We define $\mathcal{J}$ as before

$$B(v,w) = \langle v, x\cdot w\rangle^{\sigma_0}.$$

$\mathcal{J}$ is conjugate linear and $(\mathfrak{g}_0, K)$-invariant. If we try to proceed like the previous chapter, we will run into a problem, which is $x \notin G$. This is because G is unequal rank. The way we are going to solve this problem is by extending the representation to the extended group ${}^{\gamma}G$ (Definition 4.4.4).

By Lemma 5.2.6 and Clifford Theory, $(\pi, J(\Lambda, \nu))$ can be extended to two distinct $(\mathfrak{g}_0, {}^{\gamma}K)$-modules, denoted π_1 and π_2. The action of γ differs by a sign between the two extensions

$$\pi_1(\gamma) = -\pi_2(\gamma).$$

We will prove some properties of the extended modules.

Lemma 5.2.7. *If $J(\Lambda, \nu)$ admits a c-form, then its extension also admits a c-form.*

Proof. The extended module π_1 admitting a c-Hermitian form is equivalent to

$$\pi_1 \cong [\pi_1]^{h,\sigma_c}$$

It is easy to see that $[\pi_1]^{h,\sigma_c}$ is also an extension of J. Because there are only two extensions of J, we will prove the above isomorphism by contradiction. We will show

$$[\pi_1]^{h,\sigma_c} \ncong \pi_2$$

Suppose there exists such an isomorphism

$$\psi : \pi_2 \to [\pi_1]^{h,\sigma_c}$$

$$[\pi_1]^{h,\sigma_c}(\gamma) \cdot \psi(v) = \psi(\pi_2(\gamma)v) \tag{5.2}$$

Let $v \in J$ such that $\pi_1(\gamma)v = cv$ some $c \in \mathbb{C}^*$. Such vector exists because γ acts as an isomorphism $D : J \to J^{\gamma}$ and D is of finite order.

By definition, $\pi_2(\gamma)v = -cv$. We also know that $\psi(v)(v) \neq 0$ because the c-Hermitian form on J is positive definite.

Equation 5.2 left hand side evaluated on v is

$$[\pi_1]^{h,\sigma_c}(\gamma) \cdot \psi(v)(v) = \psi(v)(\pi_1(\gamma^{-1})v) = \psi(v)(c^{-1}v) = \overline{c^{-1}}\psi(v)(v)$$

The right hand side evaluated on v equals to

$$-c\psi(v)(v)$$

$-c = \overline{c^{-1}}$ if and only if $|c| = -1$. This contradicts the fact that $|c| \geq 0$ for all $c \in \mathbb{C}$. Therefore $\pi_1 \cong \pi_1^{h,\sigma_c}$. □

Lemma 5.2.8. *The extended module J_1 is Hermitian.*

This is an immediate consequence of Lemma 5.2.7 and Equation 2.1 and the fact that θ and γ are inner to each other.

Theorem 5.2.9. *Let G be an unequal rank real reductive algebraic group (Definition 2.4.4) and $(\pi, J(\Lambda, \nu))$ be an irreducible admissible Langlands quotient module. Further assume that ν is real and $J(\Lambda, \nu)$ is Hermitian. Then we have the following equation:*

$$\delta(\pi) = \varepsilon(\pi)\chi_\pi(x^2)\kappa(\pi)$$

where $x \in {}^\gamma G$ is the strong real form given by G, and

$$\kappa(\pi) = \begin{cases} 1 & \pi_1 \text{ is self-dual} \\ -1 & \pi_1 \text{ is not self-dual} \end{cases}$$

Proof. Following the unequal rank part of the proof of Theorem 4.4.2 located after Proposition 4.4.2 we arrive at

$$\varepsilon(\pi)\delta(\pi) = \operatorname{sgn}\left(\frac{\zeta^{-1}\langle x\cdot\mathcal{J}(v),\mathcal{J}(w)\rangle^c}{\overline{\zeta^{-1}\langle x\cdot v,w\rangle^c}}\right) = \operatorname{sgn}(\zeta^{-2})\operatorname{sgn}\left(\frac{\langle x\cdot\mathcal{J}(v),\mathcal{J}(w)\rangle^c}{\overline{\langle x\cdot v,w\rangle^c}}\right). \tag{5.3}$$

If the extended module J_1 is self-dual, then B is invariant under the action of x. Therefore

$$\langle v,\mathcal{J}(x\cdot w)\rangle^{\sigma_0} = B(v,x\cdot w) = B(x^{-1}\cdot v,w) = \langle x^{-1}\cdot v,\mathcal{J}(w)\rangle^{\sigma_0} = \langle v,x\cdot\mathcal{J}(w)\rangle^{\sigma_0}$$

implies

$$\mathcal{J}(x\cdot v) = x\cdot\mathcal{J}(v) \quad \forall v\in J(\Lambda,\nu)$$

If the extended module J_1 is not self-dual, then $J_1^\vee \cong J_2$ since $J_1^\vee$ is easily proven to be an extension of J. We can take the isomorphism $\psi:\pi_2\cong\pi_1^\vee$ and it will serve as an $(\mathfrak{g}_0,K)$-invariant bilinear form on J:

$$B(v,w) = \psi(v)(w) \quad \forall v,w\in J.$$

The map ψ being an intertwiner means

$$\psi(\pi_2(x)v) = \pi_1^\vee(x)\psi(v).$$

This together with the fact that $x = x_1\gamma$ implies

$$B(x\cdot v,w) = B(\pi_1(x)v,w) = \psi(\pi_1(x)v)(w) = \psi(-\pi_2(x)v)(w)$$

$$= -\pi_1^\vee(x)\psi(v)(w) = -\psi(v)(\pi_1(x^{-1})w) = -B(v,x\cdot w)$$

We define an index κ that takes the value 1 if J_1 is self-dual and -1 if J_1 is not self-dual. Then

$$B(x\cdot v,w) = \kappa(\pi)B(v,x^{-1}\cdot w)$$

This implies

$$x \cdot \mathcal{J}(v) = \kappa(\pi)\mathcal{J}(x \cdot v)$$

Equation 5.3 becomes

$$\begin{aligned}\varepsilon(\pi)\delta(\pi) &= \mathrm{sgn}(\zeta^{-2})\mathrm{sgn}\left(\frac{\langle \kappa(\pi)\mathcal{J}(x \cdot v), \mathcal{J}(w)\rangle^c}{\overline{\langle x \cdot v, w\rangle^c}}\right) \\ &= \chi_\pi(x^2)\kappa(\pi)\mathrm{sgn}\left(\frac{\langle \mathcal{J}(w), \mathcal{J}(w)\rangle^c}{\overline{\langle w, w\rangle^c}}\right)\end{aligned}$$

By Lemma 5.2.4, $\mathcal{J}$ sends LKT to LKT. Since the c-form is positive-definite on all LKT's, the theorem easily follows. □

5.2.2.2 Non-Hermitian Modules

Suppose $J(\Lambda, \nu)$ is not Hermitian. By Lemma 5.2.6, we know that $J^\gamma \ncong J$. This means we cannot extend J to a $(\mathfrak{g}_0, {}^\gamma K)$-module. Instead we can induce J to an irreducible $(\mathfrak{g}_0, {}^\gamma K)$-module, denoted $\widetilde{J}$. We will proceed by discussing properties of the induced module $\widetilde{J}$.

Lemma 5.2.10. *If J has real infinitesimal character then $\widetilde{J}$ has real infinitesimal character.*

Proof. The module $\widetilde{J}$ restricted to $(\mathfrak{g}_0, K)$ splits into two modules $\widetilde{J} = J + J^\gamma$. The action of $\mathfrak{a}_0$ on J is by the real valued character ν. We will show that ν^γ is again real valued. By definition θ acts on $\mathfrak{a}_0$ by -1 and γ is inner to θ. I.e., $\theta = \mathrm{Ad}(x_0) \circ \gamma$ where $x_0 \in H_f$ by the discussion in [7, P80]. Therefore

$$\nu^\gamma(X) = \nu(\gamma(X)) = \nu(x_0\theta(X)x_0^{-1}) = \nu(-X) = -\nu(X).$$

Clearly ν^γ is real valued on $\mathfrak{a}_0$. □

Proposition 5.2.11. *[7, Proposition 12.7] Suppose $\widetilde{J}$ is an irreducible $(\mathfrak{g}_0, {}^\gamma K)$-module of real infinitesimal character. Then $\widetilde{J}$ admits a non-degenerate c-invariant Hermitian form that is unique up to a real scalar multiple. It can be chosen to be positive-definite on the lowest ${}^\gamma K$-types of $\widetilde{J}$.*

Proposition 5.2.12. *The extended module $\widetilde{J}$ is Hermitian and self-dual.*

Proof. The twist of $\widetilde{J}$ is $\mathrm{Ind} J^\gamma$ therefore isomorphic to $\widetilde{J}$. By Proposition 5.2.11 $\widetilde{J} \cong \widetilde{J}^{h,\sigma_c}$. Therefore $\widetilde{J} \cong \widetilde{J}^{h,\sigma_0}$.

It is elementary to prove that $\widetilde{J}$ is self-conjugate given that J is self-conjugate. It is a consequence that $\widetilde{J}$ is self-dual. □

Proposition 5.2.13. *Assume G and $\widetilde{J}$ satisfy the conditions in the beginning of this section. Fix a strong real form $x \in {}^\gamma K \backslash K$ and $x^2 \in Z(K)$ acts on J as a scalar ζ^2, x^2 also acts on J^γ by ζ^2. Fix a c-invariant Hermitian form $\langle , \rangle^c$ on $\widetilde{J}$ which is positive-definite on the lowest ${}^\gamma K$-types. If $\widetilde{J}$ admits an invariant Hermitian form, then we can define $\langle , \rangle^0$*

$$\langle \widetilde{v}, \widetilde{w} \rangle^0 = \zeta^{-1} \langle x \cdot \widetilde{v}, \widetilde{w} \rangle^c = \zeta \langle \widetilde{v}, x \cdot \widetilde{w} \rangle^c$$

and it is a $(\mathfrak{g}_0, {}^\gamma K)$-invariant Hermitian form on $\widetilde{J}$.

Proof. This is again a consequence of Proposition 4.2.2 with λ replaced by x. The verification of $\langle , \rangle^0$ being an ordinary invariant Hermitian form is elementary and won't be presented here. □

Now we have shown that all the good properties that we want and J does not have are possessed by $\widetilde{J}$. Also $\widetilde{J}$ inherited the good aspects of J. For example, having

a c-invariant Hermitian form etc. This enables us to use the previous arguments on $\widetilde{J}$ to obtain a formula for $\delta(\widetilde{\pi})$.

Theorem 5.2.14. *Suppose $(\widetilde{\pi}, \widetilde{J})$ is the the induced module of J with J satisfying all the conditions we set in this section. Then*

$$\delta(\widetilde{\pi}) = \varepsilon(\widetilde{\pi})\chi_{\widetilde{\pi}}(x^2).$$

The proof of this Theorem is the same as that of Theorem 5.2.5 with J replaced by $\widetilde{J}$.

Let's come back to our main object of concern here $\delta(\pi)$. Lemma 4.5.4 shows that $\delta(\pi) = \delta(\widetilde{\pi})$; the proof applies in the infinite-dimensional case.

Theorem 5.2.15. *Let G be a real reductive algebraic group which is unequal rank, $(\pi, J(\Lambda, \nu))$ be an irreducible admissible $(\mathfrak{g}_0, K)$-module that is not Hermitian. Furthermore, J is self-conjugate and ν is real (i.e., J has real infinitesimal character). Then*

$$\delta(\pi) = \varepsilon(\widetilde{\pi})\chi_{\pi}(x^2)$$

where $\widetilde{\pi} = Ind_{(\mathfrak{g}_0, K)}^{(\mathfrak{g}_0, {}^{\gamma}K)}\pi$ and $x \in {}^{\gamma}K\backslash K$ is a strong real form given by G.

This theorem is a direct corollary of Theorem 5.2.14 and Lemma 4.5.4, with

the additional observation that $\chi_\pi(x^2) = \chi_{\widetilde{\pi}}(x^2)$.

Appendix A: Kac Classification and Formulas for Real Forms

A.1 Introduction

In this appendix, our goal is to give formulas for real forms of some unequal rank simple groups. We do this by revisiting the Kac classification, spelling out some of the theories behind the classification and give computable formulas for real forms (Theorem A.5.4). Some examples of calculating using these formulas are given in Appendix A.5.

First, we fix some notations for this Appendix. $\mathbb{G}$ denotes a complex simple adjoint Lie group of type A_{2n-1}, or D_n, or E_6, or F_4. We consider only these types because their Dynkin diagrams have non-trivial diagram automorphisms. We left out type A_{2n} because the quotient root system by a Cartan involution is possibly non-reduced. Let G denote a group of real points of $\mathbb{G}$ (Definition 2.4.3). G could be unequal rank (Definition 4.4.1). Let $\mathfrak{g}_0$ denote the Lie algebra of G, $\mathfrak{g}$ denotes the Lie algebra of $\mathbb{G}$. Consequently, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Let θ be the Cartan involution of $\mathbb{G}$ corresponding to G, γ the distinguished involution in the inner class of θ. For the definition of γ see Section 4.4.0.2. H_f denotes the fundamental Cartan (Section 4.4.0.2) of G, let $\mathbb{H}_f$ be a maximal torus of $\mathbb{G}$ such that $H_f = \mathbb{H}_f^{\sigma_0}$ (σ_0 is defined in Section 2.4). Let $T = H_f^\gamma$ and $\mathbb{T} = \mathbb{H}_f^\gamma$.

The following definition of real form is what we will use in this appendix.

Definition A.1.1. A *real form* of $\mathbb{G}$ in the inner class of γ (or θ) is an equivalence class of elements $x \in \mathbb{G}^\gamma$ (Definition 4.4.4), satisfying $x \notin \mathbb{G}$, and $x^2 \in Z(\mathbb{G})$. The equivalence is by conjugation by $\mathbb{G}$.

By the definition of γ (Definition 4.4.3), $\mathbb{H}_f$ is γ stable. The definition of $\mathbb{T}$ implies that γ fixes every element in $\mathbb{T}$. We define the following.

Definition A.1.2. Let

$$\mathbb{H}^\gamma = \mathbb{H} \rtimes \{1, \gamma\}$$

and

$$\mathbb{T}^\gamma = \mathbb{T} \times \{1, \gamma\} = \mathbb{T} \cup \mathbb{T}\gamma$$

The second definition is a direct product because γ acts on $\mathbb{T}$ trivially.

We give an outline of this appendix:

1. We prove that every real form is conjugate to an element in $\mathbb{T}\gamma$.

2. We consider the universal cover of $\mathbb{T}\gamma$. Two elements of $\mathbb{T}\delta$ are conjugate by W_γ (Definition A.2.1) if and only if some lifts of them onto the universal cover are conjugate by $\widetilde{W_\gamma}$ (Definition A.3.2).

3. Find the affine Weyl group in the extended affine Weyl group $\widetilde{W_\gamma}$, thus determining the fundamental alcove (Section A.4).

4. Describe elements in the fundamental alcove that parametrize the real forms (Lemma A.4.6).

5. Give the classification of real forms in terms of labels of Kac diagrams (Theorem A.5.3).

A.2 First properties of strong real forms

Let $\mathbb{A} := \mathbb{H}_f^{-\gamma} = \{h \in \mathbb{H}_f | \gamma(h) = h^{-1}\}$. The identity component of $H_f^{-\gamma} := \{h \in H_f | \gamma(h) = h^{-1}\}$ is defined to be A. Then

$$H_f = TA, \qquad \mathbb{H}_f = \mathbb{T}\mathbb{A}$$

Lemma A.2.1. *After conjugation by $\mathbb{G}$, we may assume*

$$x = x_0\gamma, \quad x_0 \in \mathbb{T}$$

Proof. We first show that if $x = x_0\gamma$ is a real form, then x_0 has to be a semi-simple element in $\mathbb{G}$.

Let $\theta = \mathrm{Ad}(x)$. Keep in mind the fact:

$$\mathbb{H}_f = Cent_{\mathbb{G}}(\mathbb{T})$$

and γ fixes every element in $\mathbb{T}$.

For an element $h \in \mathbb{H}_f$, it is easy to see that $\theta(h) \in \mathbb{H}_f$ because for all $t \in \mathbb{T}$

$$\theta(h)t = \theta(ht) = \theta(th) = t\theta(h)$$

Therefore $\theta(h) \in Z_{\mathbb{G}}(\mathbb{T}) = \mathbb{H}_f$.

On the other hand:

$$\theta(h) = x_0\delta(h)x_0^{-1} = x_0hx_0^{-1} \quad \Rightarrow \quad h = \theta(x_0)\theta(h)\theta(x_0)^{-1} \ \ \forall h \in \mathbb{H}_f$$

These facts combined show that $\theta(x_0)^{-1} \in Norm_{\mathbb{G}}(\mathbb{H}_f)$, therefore $\theta(x_0^{-1})$ is of finite order in $Norm_{\mathbb{G}}(\mathbb{H}_f)/\mathbb{H}_f$, i.e., a finite power of $\theta(x_0^{-1})$ is semi-simple, therefore $\theta(x_0^{-1})$ is semi-simple, thus x_0^{-1} is semi-simple; see [11, Chapter 2] for omitted arguments.

We claim that there exists a γ-stable Cartan $\mathbb{H}_1$ of G such that x_0 is an element of $\mathbb{H}_1$. We prove this claim now. Let $\mathbb{L}$ be the identity component of $\mathbb{M} = Cent_{\mathbb{G}}(x_0)$. Using simple algebra, it is easy to show $\gamma(\mathbb{L})$ is the identity component of $Cent_{\mathbb{G}}(\gamma(x_0))$. We already know $\gamma(x_0) = x_0^{-1}x^2$, $x^2 \in Z(\mathbb{G})$. This gives:

$$\gamma(\mathbb{M}) = Cent_{\mathbb{G}}(x_0^{-1}x^2) = Cent_{\mathbb{G}}(x_0) = \mathbb{M}$$

It is not hard to see that the Levi $\mathbb{L}$ is γ-stable, so we take the γ-stable Cartan $\mathbb{H}_1$ in $\mathbb{M}$, and then x_0 has to be contained in $\mathbb{H}_1$.

Write $\mathbb{H}_1 = \mathbb{T}_1\mathbb{A}_1$ where $\mathbb{T}_1 := (\mathbb{H}_1^{\gamma})_0$ and $\mathbb{A}_1 := (\mathbb{H}_1^{-\gamma})_0$. We next show that x_0 can be conjugated to an element in $\mathbb{T}_1$.

$x_0 \in \mathbb{H}_1$ can be written as $x_0 = ta$, since $\mathbb{A}_1^2 = \mathbb{A}_1$ (see the remark right after this lemma); we take $b \in \mathbb{A}_1$ so that $b^2 = a^{-1}$, conjugate x with b:

$$bxb^{-1} = bta\delta b^{-1} = bta\gamma(b^{-1})\delta = tab^2\delta = taa^{-1}\delta = t\delta$$

Since x_0 can be chosen to be in $\mathbb{T}_1$, which is a torus in the identity component of $\mathbb{K} := \mathbb{G}^{\gamma}$, x_0 can be then conjugated to an element in $\mathbb{T}$ by $k \in \mathbb{K}$. Use k to conjugate x:

$$kxk^{-1} = kx_0\delta k^{-1} = kx_0\gamma(k^{-1})\delta = kx_0k^{-1}\delta$$

This completes the proof of the lemma. □

Remark 14. Given γ a Cartan involution and $\mathbb{H}$ any γ stable Cartan in G, we may write $\mathbb{H} = (\mathbb{C}^*)^a \times (\mathbb{C}^*)^b \times (\mathbb{C}^* \times \mathbb{C}^*)^c$, where γ acts trivially on the first a factors, by inverse on the next b factors, and $\gamma(s,t) = (t,s)$ on each of the last c terms.

$$\gamma(z_1, \cdots, z_a, w_1, \cdots, w_b, s_1, t_1, \cdots, s_c, t_c)$$
$$= (z_1, \cdots, z_a, w_1^{-1}, \cdots, w_b^{-1}, t_1, s_1, \cdots, t_c, s_c)$$

$$\mathbb{H}^\gamma = (z_1, \cdots, z_a, \pm 1, \cdots, \pm 1, t_1, t_1, t_2, t_2, \cdots, t_c, t_c)$$

$$\mathbb{H}^{-\gamma} = (\pm 1, \cdots, \pm 1, w_1, \cdots, w_b, t_1^{-1}, t_1, t_2^{-1}, t_2 \cdots, t_c^{-1}, t_c)$$

Take the identity components:

$$\mathbb{T} = (z_1, \cdots, z_a, 1, \cdots, 1, t_1, t_1, t_2, t_2, \cdots, t_c, t_c)$$

$$\mathbb{A} = (1, \cdots, 1, w_1, \cdots, w_b, t_1^{-1}, t_1, t_2^{-1}, t_2 \cdots, t_c^{-1}, t_c)$$

Because $\mathbb{A}$ is a connected torus, it obvious that $\mathbb{A}^2 = \mathbb{A}$.

Definition A.2.1. Define the set of inequivalent normalizers W_γ of $\mathbb{T}\gamma$ in $\mathbb{G}$ to be:

$$W_\gamma = Norm_{\mathbb{G}}(\mathbb{T}\gamma)/Cent_{\mathbb{G}}(\mathbb{T}\gamma)$$

Lemma A.2.2. *The real forms in the inner class of θ are parametrized by elements $x \in \mathbb{T}\gamma$ such that $x^2 \in Z(\mathbb{G})$, modulo the action of W_γ.*

In order to express x using the root system $\Delta(\mathbb{G}, \mathbb{H}_f)$, it is natural to seek the help of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$. In the next subsection, we are going to introduce the universal cover E of $\mathbb{T}\gamma$, which is isomorphic to $\mathfrak{t}$ once we picked an origin. The problem of classifying real forms will be moved up to this affine space E.

A.3 The universal cover

Definition A.3.1. [12, Chapter 1.1] An *affine space* E over a field F is a set on which a F-vector space V acts faithfully and transitively. The elements of V are called *translations* of E, and the effect of a translation $v \in V$ on $x \in E$ is written $x + v$. If $y = x + v$ we write $v = y - x$.

Let $\pi : E \to \mathbb{T}\gamma$ be the universal cover. It is an affine space with translations $\mathfrak{t}$. Recall the construction of universal cover. We pick a base point, in this case let's say γ, the fibers over $y \in \mathbb{T}\gamma$ in E are homotopy classes of paths from γ to y.

$$E = \{x : [0,1] \to \mathbb{T}\gamma | x(0) = \gamma\} / \sim$$

$$\pi(x) = x(1), \quad \forall x \in E$$

The action of $\mathfrak{t}$ on E is as follows: for $X \in \mathfrak{t}$

$$(X + a)(r) = \exp(2\pi i r X) \cdot a(r), \quad \forall r \in [0,1]$$

Pick an element $\tilde{\gamma} \in E$, where $\tilde{\gamma}(r) = \gamma$, $\forall r \in [0,1]$. This gives an isomorphism

$$\mathfrak{t} \cong E$$

$$X \mapsto X + \tilde{\gamma}$$

Moreover, $\pi(X + \tilde{\gamma}) = \exp(2\pi i X)\delta$ for an arbitrary element $X + \tilde{\gamma} \in E$.

Remark 15. The following is true by a well known property for universal covers: for $f : \mathbb{T}\gamma \to \mathbb{T}\gamma$ a differentiable map, and for any points $x, y \in E$ such that $f(\pi(x)) = \pi(y)$ there exists a unique differentiable map $\tilde{f} : E \to E$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}\gamma & \xrightarrow{f} & \mathbb{T}\gamma \end{array}$$

commutes and $\tilde{f}(x) = y$. In this case we say $\tilde{f}$ *covers* f. The group of covers of the identity map is isomorphic to the fundamental group $\pi_1(\mathbb{T}\gamma)$.

Definition A.3.2.

$$\widetilde{W_\gamma} = \{\tilde{w} : E \to E | \tilde{w} \text{ covers some } w \in W_\gamma\}$$

Lemma A.3.1. *The real forms in the inner class of γ are parametrized by elements $x \in E$ such that $\pi(x)^2 \in Z(\mathbb{G})$, modulo the action of $\widetilde{W_\gamma}$*

Proof. It is clear that $w(t_1\gamma) = t_2\gamma$ if and only if for any x_1, x_2 where $\pi(x_i) = t_i\gamma$ there is a lift of w, call it $\tilde{w}$, such that $\tilde{w}(x_1) = x_2$. In other words, $\pi : E \to \mathbb{T}\gamma$ factors to a bijection:

$$E/\widetilde{W_\gamma} \leftrightarrow \mathbb{T}\gamma/W_\gamma$$

This lemma is clearly equivalent to Lemma A.2.2 □

A.4 The extended affine Weyl group $\widetilde{W_\gamma}$

Let us first identify the elements in $\widetilde{W_\gamma}$ that act by an affine Weyl group. More specifically, the ones that fixes $\tilde{\gamma}$ and the ones that acts by a translation on E.

1. elements that fix $\tilde{\gamma}$:

For an element $\tilde{w} \in \widetilde{W_\gamma}$ to fix $\tilde{\gamma}$, it has to be a lift from $w_g \in W_\gamma$ which fixes γ, the notation w_g means $w_g(t\gamma) = gt\gamma g^{-1}$. It's easy to see, such g has to be in $\mathbb{K}$.

So $W_\mathbb{K} := Norm_\mathbb{K}(\mathbb{T})/Cent_\mathbb{K}(\mathbb{T})$ is in bijection with the set of elements in $\widetilde{W_\gamma}$ that fix $\tilde{\gamma}$.

2. elements that acts by translation:

Translations on E are of the form: $X + (Y + \tilde{\gamma})$, where $Y + \tilde{\gamma}$ is an arbitrary element in E. Suppose translation by $X \in \mathfrak{t}$ is an element in $\widetilde{W_\gamma}$, that means it is a lift of some element $w_g \in W_\gamma$, i.e.,

$$gt\gamma g^{-1} = \pi(X + Y_t + \tilde{\gamma}) = \exp(2\pi i X)t\gamma \quad \forall t \in \mathbb{T}$$

where Y_t is some element in $\mathfrak{t}$ such that $\exp(2\pi i Y_t) = t$. It is not hard to see that $\exp(2\pi i X)$ is in $\mathbb{A} \cap \mathbb{T}$: let $s = \exp(2\pi i X) \in \mathbb{T}$

$$g\gamma(g^{-1}) = s \Rightarrow \gamma(g^{-1}) = g^{-1}s$$

$$gt\gamma g^{-1} = gt\gamma(g^{-1})\gamma = gtg^{-1}s\gamma = sgtg^{-1}\gamma = st\gamma \Rightarrow g \in Cent_\mathbb{G}(\mathbb{T}) = \mathbb{H}_f$$

Since $\mathbb{H}_f = \mathbb{T}\mathbb{A}$, we can write $g = t_g a_g$ with $t_g \in \mathbb{T}, a_g \in \mathbb{A}$.

$$\gamma(g^{-1}) = \gamma(a_g^{-1}t_g^{-1}) = a_g t_g^{-1} = g^{-1}s = a_g^{-1}t_g^{-1}s \Rightarrow a_g^2 = s$$

In fact, each element in $\mathbb{A} \cap \mathbb{T}$ lifts to a translation on E, just take the square root of that element in $\mathbb{A}$ to be the g in w_g, since $\mathbb{A}$ and $\mathbb{T}$ are both closed under taking square root.

Therefore the set of pre-image of $\mathbb{A} \cap \mathbb{T}$ under the map $\exp(2\pi i -)$ is the set of translations on E.

We now describe these elements in terms of Weyl groups and roots/co-roots. First, let's fix some notations. Let $\Delta(\mathbb{G}, \mathbb{H}_f)$ be the set of roots of $\mathbb{H}_f$ in $\mathbb{G}$, and $\Delta^\vee(\mathbb{G}, \mathbb{H}_f)$ be the set of co-roots. Let $R(\mathbb{G}, \mathbb{H}_f) = \langle \Delta(\mathbb{G}, \mathbb{H}_f) \rangle$ be the root lattice, and $R^\vee(\mathbb{G}, \mathbb{H}_f) = \mathbb{Z}\langle \Delta^\vee(\mathbb{G}, \mathbb{H}_f) \rangle$ be the co-root lattice.

The following lemma will be frequently used.

Lemma A.4.1. *[13, Lemma 2.5] Let $\Delta_{cx}(\mathbb{G}, \mathbb{H}_f)$ be the subset of complex roots in $\Delta(\mathbb{G}, \mathbb{H}_f)$. Suppose $\tau \in \Delta_{cx}(\mathbb{G}, \mathbb{H}_f)$, one of the following conditions hold:*

(1) $\langle \tau, \gamma(\tau^\vee) \rangle = 0$, and $\tau^\vee + \gamma\tau^\vee = \beta^\vee$ where $\beta = \tau|_T \in \Delta(\mathbb{K}, \mathbb{T})$,

(2) $\langle \tau, \gamma(\tau^\vee) \rangle = -1$, and $\tau^\vee + \gamma\tau^\vee = \beta^\vee$ where $\beta = \tau + \gamma\tau \in \Delta_{nc}$

In (1) we have implicitly used the inclusions $\beta^\vee \in \Delta^\vee(\mathbb{K}, \mathbb{T}) \subset X_*(\mathbb{T}) \subset X_*(\mathbb{H}_f) = R^\vee(\mathbb{G}, \mathbb{H}_f)$.

Lemma A.4.2.

$$\{X \in \mathfrak{t} | \exp(2\pi i X) \in \mathbb{A} \cap \mathbb{T}\} = R^\vee_{\mathbb{K}} \cup \{\frac{1}{2}\alpha^\vee | \alpha \in R_{\mathbb{K}} \text{ short } \}$$

where $R_{\mathbb{K}}$ is the root lattice of $\Delta(\mathbb{K}, \mathbb{T})$ and $R^\vee_{\mathbb{K}}$ is the coroot lattice.

Proof. It is clear that

$$\{X \in \mathfrak{t} | \exp(2\pi i X) \in \mathbb{A} \cap \mathbb{T}\} = \{X \in \mathfrak{t} | \exp(2\pi i X) \in \mathbb{A}\}$$

because $X \in \mathfrak{t}$ implies $\exp(2\pi i X) \in \mathbb{T}$. The following inclusions are also elementary to prove.

$$\{X \in \mathfrak{t} | X \in ker(\exp)\} \subseteq \{X \in \mathfrak{t} | \exp(2\pi i X) \in \mathbb{A}\} \subseteq \{\frac{1}{2}X | X \in ker(\exp), X \in \mathfrak{t}\}$$

The assumption that $\mathbb{G}$ is adjoint allows us to identify $ker(\exp)$ with $R_{\mathbb{K}}^{\vee}$. Let

$$L = \{X \in \mathfrak{t} | \exp(2\pi i X) \in \mathbb{A}\}$$

Claim: $\gamma^{\vee} \in L \Leftrightarrow \exp(2\pi i \gamma^{\vee}) = \exp(2\pi i \mu^{\vee})$ where $\gamma(\mu^{\vee}) = -\mu^{\vee}$.

From the claim, we have $\beta^{\vee} \in L \Leftrightarrow \beta^{\vee} - \mu^{\vee} \in X_*(\mathbb{H}_f) \Leftrightarrow \beta^{\vee} - \mu^{\vee} = \tau^{\vee}$, some $\tau^{\vee} \in X_*(\mathbb{H}_f)$ ($X_*(\mathbb{H}_f)$ is just $R^{\vee}(\mathbb{G}, \mathbb{H}_f)$ by the assumption that $\mathbb{G}$ is adjoint.) Apply $(1+\gamma)$ to both sides of $\beta^{\vee} - \mu^{\vee} = \tau^{\vee}$, we get

$$(1+\gamma)\beta^{\vee} = (1+\gamma)\tau^{\vee}, \quad \tau^{\vee} \in R^{\vee}(\mathbb{G}, \mathbb{H}_f)$$

Therefore for any $\alpha^{\vee} \in R_{\mathbb{K}}^{\vee}$, $\frac{\alpha^{\vee}}{2} \in L \Leftrightarrow (1+\gamma)\frac{\alpha^{\vee}}{2} = (1+\gamma)\tau^{\vee}$, some $\tau^{\vee} \in R^{\vee}(\mathbb{G}, \mathbb{H}_f)$. I.e.,

$$\alpha^{\vee} = \tau^{\vee} + \gamma(\tau^{\vee}), \quad \tau^{\vee} \in R^{\vee}(\mathbb{G}, \mathbb{H}_f)$$

Let's see which element in $R_{\mathbb{K}}^{\vee}$ are of the form $\tau^{\vee} + \gamma\tau^{\vee}$, $\tau^{\vee} \in R^{\vee}(\mathbb{G}, \mathbb{H}_f)$. By Lemma A.4.1, we can conclude (not including type A_{2n}) for $\tau \in \Delta_{cx}(\mathbb{G}, \mathbb{H}_f)$, the coroot corresponding to $\beta = \tau|_{\mathbb{T}}$ is of such form, i.e., $\{\frac{1}{2}\beta^{\vee} | \beta \in R_{\mathbb{K}} \text{ short }\}$ is in L. This completes the proof. □

Lemma A.4.3.

$$R_{\mathbb{K}}^{\vee} \cup \{\frac{1}{2}\alpha^{\vee} | \alpha \in R_{\mathbb{K}} \textit{ short }\} = \frac{1}{l}R_{\mathbb{K}}$$

where l is the square of the length of any short root.

Proof. Because $R_{\mathbb{K}}$ only has two root lengths, for short roots $||\alpha||^2 = l$ and long root $||\beta||^2 = 2l$, we can easily calculate

$$R_{\mathbb{K}}^{\vee} = \{\frac{2\alpha}{l} | \alpha \in R_{\mathbb{K}} \text{ short}\} \cup \{\frac{2\beta}{2l} | \beta \in R_{\mathbb{K}} \text{ long}\}$$

Therefore

$$R_{\mathbb{K}}^{\vee} \cup \{\frac{1}{2}\alpha^{\vee} | \alpha \in R_{\mathbb{K}} \text{ short } \} = \{\frac{\alpha}{l} | \alpha \in R_{\mathbb{K}}\} = \frac{1}{l} R_{\mathbb{K}}$$

□

Definition A.4.1. Let W_{aff} be the group of displacements on E generated by the Weyl group W_K and the translations $t(\frac{1}{l}\alpha), \alpha \in R_{\mathbb{K}}$, so that W_{aff} is the semidirect product of W_K and $t(\frac{1}{l}R_{\mathbb{K}})$:

$$W_{aff} = W_K \ltimes t(\frac{1}{l} R_{\mathbb{K}})$$

Remark 16. The specific action of W_K on $t(\frac{1}{l}R_{\mathbb{K}})$ is:

$$s_\alpha t(\frac{1}{l}\beta) s_\alpha = t(s_\alpha(\frac{1}{l}\beta)) \qquad \forall \alpha, \beta \in R_{\mathbb{K}}$$

The usual reflection on $R_{\mathbb{K}}$ is:

$$s_\alpha(\beta) = \beta - (\beta, \alpha)\alpha^{\vee}$$

and on $R_{\mathbb{K}}^{\vee}$:

$$s_\alpha(\beta^{\vee}) = \beta^{\vee} - \frac{2(\beta, \alpha)}{(\beta, \beta)}\alpha^{\vee}$$

The reflection on E is defined as follows [12, P2]:

$$s_\alpha(x) = x - \alpha^{\vee}(x) D\alpha = x - \alpha(x) D\alpha^{\vee} = x - \alpha(x)\alpha^{\vee}$$

So for any element $x \in E$:

$$\begin{aligned} s_\alpha t(\frac{1}{l}\beta)s_\alpha(x) &= s_\alpha t(\frac{1}{l}\beta)(x - \alpha(x)\alpha^\vee) = s_\alpha(\frac{1}{l}\beta + x - \alpha(x)\alpha^\vee) \\ &= \frac{1}{l}\beta + x - \alpha(x)\alpha^\vee - \alpha(\frac{1}{l}\beta + x - \alpha(x)\alpha^\vee)\alpha^\vee \\ &= \frac{1}{l}\beta + x - \alpha(x)\alpha^\vee - \frac{1}{l}\alpha(\beta)\alpha^\vee - \alpha(x)\alpha^\vee + \alpha(x)\alpha(\alpha^\vee)\alpha^\vee \\ &= \frac{1}{l}\beta - \frac{1}{l}\alpha(\beta)\alpha^\vee - 2\alpha(x)\alpha^\vee + 2\alpha(x)\alpha^\vee + x \\ &= t(s_\alpha(\frac{1}{l}\beta))(x) \end{aligned}$$

The last step in detail:

$$\alpha(\frac{1}{l}\beta) = \begin{cases} \alpha(\beta^\vee) & \beta \text{ long} \\ \alpha(\frac{1}{2}\beta^\vee) & \beta \text{ short} \end{cases} = \begin{cases} \frac{2(\alpha,\beta)}{(\beta,\beta)} & \beta \text{ long} \\ \frac{(\alpha,\beta)}{(\beta,\beta)} & \beta \text{ short} \end{cases}$$

$$\frac{1}{l}\beta - \alpha(\frac{1}{l}\beta)\alpha^\vee = \begin{cases} \beta^\vee - \frac{2(\alpha,\beta)}{(\beta,\beta)}\alpha^\vee & \beta \text{ long} \\ \frac{1}{2}\beta^\vee - \frac{(\alpha,\beta)}{(\beta,\beta)}\alpha^\vee & \beta \text{ short} \end{cases} = \begin{cases} s_\alpha(\beta^\vee) = s_\alpha(\frac{1}{l}\beta) & \beta \text{ long} \\ \frac{1}{2}s_\alpha(\beta^\vee) = s_\alpha(\frac{1}{l}\beta) & \beta \text{ short} \end{cases}$$

Lemma A.4.4. *W_{aff} embeds in $\widetilde{W_\gamma}$.*

Proof. For elements $w_k \in W_K$, it maps to a unique lifting $\tilde{w}_k \in \widetilde{W_\gamma}$ which sends $\tilde{\gamma}$ to $\tilde{\gamma}$. For an element $t(X)$ in $\frac{1}{l}R_{\mathbb{K}}$, it is the lifting of $w_a \in W_\gamma$ where $a^2 = \exp(2\pi i X)$ (see the beginning of the section about elements that acts by translation). □

Proposition A.4.5. *The fundamental alcove Λ of W_{aff} on E is bounded by the hyperplanes of $\{\widetilde{\alpha}_0, \widetilde{\alpha}_1, \cdots, \widetilde{\alpha}_l\}$ where $\alpha_1, \cdots, \alpha_l$ are the simple roots in $\Delta(\mathbb{K}, \mathbb{T})$, and α_0 is the highest short root in $\Delta(\mathbb{K}, \mathbb{T})$.*

$$\widetilde{\alpha}_i = \begin{cases} (\alpha_0, \frac{1}{2}) & i = 0 \\ (\alpha_i, 0) & 1 \le i \le l \end{cases}$$

note that (α, c) denotes an affine functional on E: $(\alpha, c)(X + \tilde{\gamma}) = \alpha(X) + c$.

Proof. Using the well developed theory in [12, Ch1-2] about affine Weyl groups, we only need to recognize that

$$W_{aff} = W_K \ltimes \frac{1}{l}(R_{\mathbb{K}}^{\vee})^{\vee}$$

by [12], the fundamental alcove is bounded by $H_{(\alpha_i,0)}, 1 \leq i \leq l$ and $H_{(-\varphi,\frac{1}{l})}$ where φ is the highest (long) root in $R_{\mathbb{K}}^{\vee}$. The notation

$$H_{(\alpha,c)} = \{x \in E | (\alpha, c)(x) = 0\}$$

denote the hyperplane of (α, c).

$\varphi^{\vee}$ is the highest short root in $R_{\mathbb{K}}$, call that α_0. $\varphi^{\vee} = \alpha_0 \Rightarrow \alpha_0^{\vee} = \varphi$. So $H_{(-\varphi,\frac{1}{l})} = H_{(-\alpha_0^{\vee},\frac{1}{l})} = H_{(-\frac{2\alpha_0}{l},\frac{1}{l})} = H_{(-\alpha_0,\frac{1}{2})}$. The claim follows. □

Lemma A.4.6. *The real forms in the inner class of θ are parametrized by elements in the closure of the fundamental alcove $x \in \overline{\Lambda}$ such that $\pi(x)^2 \in Z(\mathbb{G})$, modulo the action of $\widetilde{W_\delta}/W_{aff}$*

A.5 Formula for strong real forms

The elements in the fundamental alcove Λ can be described using its vertices. The vertices $x_0, \cdots x_l$ satisfy:

$$\widetilde{\alpha}_j(x_i) = 0, j \neq i$$

a simple calculation gives:

$$x_i = \begin{cases} \tilde{\gamma} & i = 0 \\ \frac{\gamma_i^{\vee}}{2m_i} + \tilde{\gamma} & 1 \leq i \leq l \end{cases}$$

where $\gamma_i^\vee$ are the fundamental coweights of $\mathbb{K}$ with respect to $\mathbb{T}$ and

$$\alpha_0 = \sum_{i=1}^{l} m_i \alpha_i$$

m_i are the labels for simple roots, the default label for α_0 is $m_0 = 1$. Note that

$$\widetilde{\alpha}_i(x_i) = \frac{1}{2m_i} \quad \forall i$$

Lemma A.5.1. *We may parametrize* $\overline{\Lambda}$ *as*

$$\{\sum_{i=0}^{l} \lambda_i x_i | \sum_{i=0}^{l} \lambda_i = 1, \lambda_i \geq 0\}$$

equivalently, $\overline{\Lambda}$ *can also be parametrized by*

$$\{(a_0, \cdots, a_l) | a_i \geq 0, \sum_{i=0}^{l} m_i a_i = \frac{1}{2}\}$$

Proof. We just need to construct a bijection between the two sets:

Given $x = \sum_{i=0}^{l} \lambda_i x_i \in \overline{\Lambda}$, let

$$a_i = \widetilde{\alpha}_i(x), \forall i \Rightarrow a_i = \frac{\lambda_i}{2m_i} \geq 0, \forall i$$

$$\sum_{i=0}^{l} m_i a_i = \sum_{i=0}^{l} \frac{\lambda_i}{2} = \frac{1}{2} \sum_{i=0}^{l} \lambda_i = \frac{1}{2}$$

Given $(a_0, \cdots, a_l)$ where $a_i \geq 0$ and $\sum_{i=0}^{l} m_i a_i = \frac{1}{2}$, let

$$\lambda_i = 2m_i a_i, \forall i$$

and

$$x = \sum_{i=0}^{l} \lambda_i x_i$$

x is an element in $\overline{\Lambda}$ because $\sum_{i=0}^{l} \lambda_i = \sum_{i=0}^{l} 2m_i a_i = 1$ □

Lemma A.5.2. *Let $(a_0, \cdots, a_l)$ be the parameter for $X \in \overline{\Lambda}$, $\pi(X)^2 \in Z(\mathbb{G})$ if and only if $2a_i \in \mathbb{Z}\ \forall i$.*

Proof. X can be expressed as $X = \sum_{i=0}^{l} \lambda_i x_i$, where $\lambda_i = 2m_i a_i$.

$$\begin{aligned} X = \sum_{i=0}^{l} \lambda_i x_i &= (1 - \lambda_1 - \cdots - \lambda_l)x_0 + \lambda_1 x_1 + \cdots + \lambda_l x_l \\ &= x_0 + \lambda_1(x_1 - x_0) + \lambda_2(x_2 - x_0) + \cdots + \lambda_l(x_l - x_0) \\ &= \frac{\lambda_1}{2m_1}\gamma_1^\vee + \frac{\lambda_2}{2m_2}\gamma_2^\vee + \cdots + \frac{\lambda_l}{2m_l}\gamma_l^\vee + \tilde{\gamma} \end{aligned}$$

$$\pi(X)^2 = \exp(2\pi i \sum_{i=1}^{l} \frac{\lambda_i}{2m_i}\gamma_i^\vee)^2 = \exp(2\pi i \sum_{i=1}^{l} 2a_i\gamma_i^\vee) \in Z(\mathbb{G})$$

if and only if $\sum_{i=1}^{l} 2a_i\gamma_i^\vee \in P^\vee$ (the weight lattice) if and only if $2a_i \in \mathbb{Z}$, for $i \geq 1$. Furthermore, $2a_0 \in \mathbb{Z}$ because

$$2a_i \in \mathbb{Z}, \forall i \geq 1 \Rightarrow \frac{\lambda_i}{m_i} \in \mathbb{Z}, \forall i \geq 1 \Rightarrow \lambda_i \in \mathbb{Z}, \forall i \geq i \Rightarrow \lambda_0 = 1 - \sum_{i=1}^{l} \lambda_i \in \mathbb{Z}$$

$$\Rightarrow a_0 = 0 \text{ or } \frac{1}{2} \Rightarrow 2a_0 \in \mathbb{Z}$$

The lemma follows. □

Theorem A.5.3. *The real forms in the inner class of θ are parametrized by a node with label $m_i = 1$ in the extended Dynkin diagram with α_0 of $\Delta(\mathbb{K}, \mathbb{T})$, the points in E are of the form*

$$x_i = \begin{cases} \tilde{\gamma} & i = 0 \\ \frac{\gamma_i^\vee}{2} + \tilde{\gamma} & i \neq 0 \end{cases}$$

the real forms are:

$$x = \pi(x_i) = \begin{cases} \gamma & i = 0 \\ \exp(\pi i \gamma_i^\vee)\gamma & i \neq 0 \end{cases}$$

modulo the action of $\widetilde{W_\gamma}/W_{aff}$.

Proof. By the previous lemma, strong real forms are parametrized by

$$\{(a_0, \cdots, a_l) | a_i \geq 0, 2a_i \in \mathbb{Z}, \sum_{i=0}^{l} m_i a_i = \frac{1}{2}\}$$

Since $m_i \geq 1$ and $m_i \in \mathbb{Z}$, the only possible value for m_i to take so that the condition holds is when $m_i = 1$, $a_i = \frac{1}{2}$ and $a_j = 0, \forall j \neq i$. Therefore $\lambda_i = 1$, $\lambda_j = 0, \forall j \neq i$.

$$x = \sum_{i=0}^{l} \lambda_i x_i = x_i$$

□

In order to make it more convenient for calculation, we want to express strong real forms in term of information from $\Delta(\mathbb{G}, \mathbb{H}_f)$.

Let $\{\xi_1, \cdots, \xi_n\}$ be the set of simple roots in $\Delta(\mathbb{G}, \mathbb{H}_f)$, such that after arranging them we have $\{\xi_1|_{\mathbb{T}}, \cdots, \xi_l|_{\mathbb{T}}\}$ as the set of simple roots in $\Delta(\mathbb{K}, \mathbb{T})$. Let $\{\gamma_1^\vee, \cdots, \gamma_l^\vee\}$ be the fundamental coweights of $\mathbb{G}$ with respect to $\mathbb{H}_f$.

Claim 2.

$$\eta_i^\vee = \begin{cases} \gamma_i^\vee + \gamma(\gamma_i^\vee) & \gamma(\gamma_i^\vee) \neq \gamma_i^\vee \\ \gamma_i^\vee & \gamma(\gamma_i^\vee) = \gamma_i^\vee \end{cases}$$

are the fundamental coweights of $\mathbb{K}$ with respect to $\mathbb{T}$.

Proof. Let $\alpha_i = \xi_i|_{\mathbb{T}}$, $i \leq l$.

$$\alpha_i(\eta_j^\vee) = \begin{cases} \xi_i|_T(\gamma_j^\vee + \gamma(\gamma_j^\vee)) = \xi_i(\gamma_j^\vee + \gamma(\gamma_j^\vee)) & \gamma(\gamma_j^\vee) \neq \gamma_j^\vee \\ \xi_i|_T(\gamma_j^\vee) = \xi_i(\gamma_j^\vee) & \gamma(\gamma_j^\vee) = \gamma_j^\vee \end{cases}$$

We know that γ acts on the set of simple roots $\{\xi_i\}$, therefore it acts on the set of fundamental coweights $\{\gamma_i^\vee\}$. Moreover, if γ doesn't fix $\xi_i, i \leq l$, then it sends it to $\xi_j, j \geq l$. So if $j \neq i$ and $\gamma(\gamma_j^\vee) \neq \gamma_j^\vee$ then $\gamma(\gamma_j^\vee) = \gamma_k^\vee$ with $k \geq l$ and $k \neq i$. Simple reasonings like this would show that

$$\alpha_i(\eta_j^\vee) = \delta_{ij}$$

therefore completing the proof. □

Theorem A.5.3 in terms of the root information of $\mathbb{G}$ is:

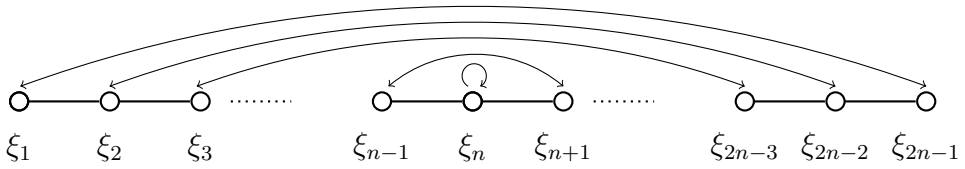
Theorem A.5.4. *The real forms in the inner class of γ are parametrized by a node with label $m_i = 1$ in the extended Dynkin diagram with α_0 being the highest short root of $\Delta(\mathbb{K}, \mathbb{T})$, the real forms are:*

$$x = \pi(x_i) = \begin{cases} \gamma & i = 0 \\ \exp(\pi i(\gamma_i^\vee + \gamma(\gamma_i^\vee)))\gamma & i \neq 0, \gamma(\gamma_i^\vee) \neq \gamma_i^\vee \\ \exp(\pi i \gamma_i^\vee)\gamma & i \neq 0, \gamma(\gamma_i^\vee) = \gamma_i^\vee \end{cases}$$

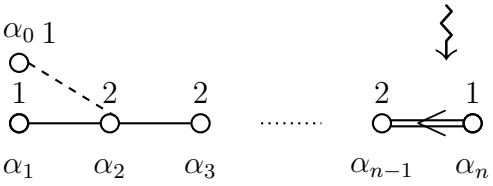
modulo the action of $\widetilde{W}_\gamma / W_{aff}$.

The following is a list of real forms, Dynkin diagram of $\mathbb{G}$ with respect to $\mathbb{H}_f$ and extended dynkin diagram of $\mathbb{K}$ with respect to $\mathbb{T}$. The symbol $\rightsquigarrow$ means fold the diagram according the the non-trivial diagram automorphism indicated by the arrows.

A_{2n-1}

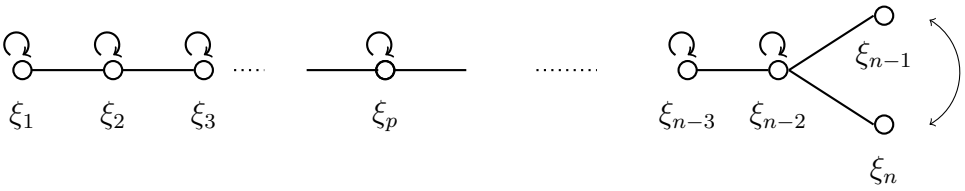


C_n

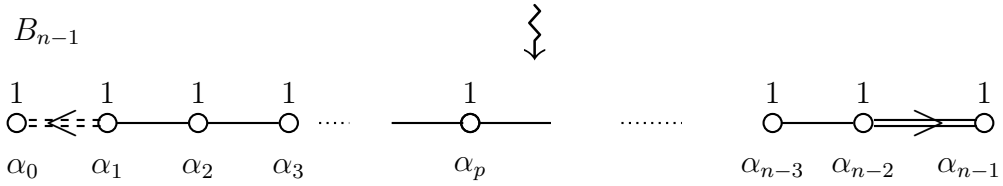


$$x = \begin{cases} \gamma & \text{pick } \alpha_0, \quad \text{represent } \mathfrak{sl}_n(\mathbb{H}) \\ \exp(\pi i(\gamma_1^\vee + \gamma_{2n-1}^\vee))\gamma & \text{pick } \alpha_1, \quad \text{represent } \mathfrak{sl}_n(\mathbb{H}) \\ \exp(\pi i \gamma_n^\vee)\gamma & \text{pick } \alpha_n, \quad \text{represent } \mathfrak{sl}_{2n}(\mathbb{R}) \end{cases}$$

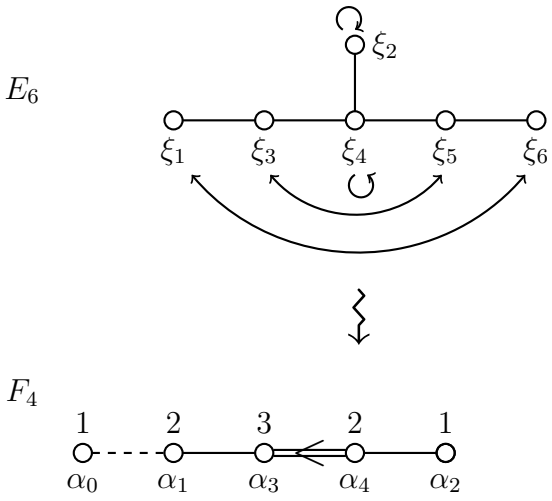
D_n



B_{n-1}



$$x = \begin{cases} \gamma & \text{pick } \alpha_0, \quad \text{represent } \mathfrak{so}_{1,2n-1} \\ \exp(\pi i \gamma_p^\vee)\gamma & \text{pick } \alpha_p, 1 \le p \le n-1, \quad \text{represent } \mathfrak{so}_{2p+1,2(l-p)-1} \\ \exp(\pi i(\gamma_{n-1}^\vee + \gamma_n^\vee))\gamma & \text{pick } \alpha_{n-1}, \quad \text{represent } \mathfrak{so}_{2n-1,1} \end{cases}$$



$$x = \begin{cases} \gamma & \text{pick } \alpha_0, \quad \text{represent the split real form} \\ \exp(\pi i \gamma_2^\vee)\gamma & \text{pick } \alpha_2, \quad \text{represent } E_6(F_4) \end{cases}$$

Appendix B: Correction of Table in [1]

There is no known list of errata for the book "Lie Groups and Algebraic Groups" by Onishchik and Vinberg [1]. This appendix is a correction to the table of indicators on page 292 of this book. We will give the correction of the table, followed by detailed calculation of relevant indicators. Also, the formula on page 291 of [1] is a special case of Theorem 4.4.2.

$\mathfrak{g}$	$\delta(\rho(\Lambda))$
$\mathfrak{su}_{k,2p-k}$	$(-1)^{(k+1)p\Lambda_{p-1}}$
$\mathfrak{u}_l^*(\mathbb{H})$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2\lfloor l/2\rfloor-1}}$
$\mathfrak{sl}_p(\mathbb{H})$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2p-1}}$
$\mathfrak{so}_{2k-1,2(l-k)+1}$	$(-1)^{((k+1)+(l-1)(l-2)/2)(\Lambda_{l-1}+\Lambda_l)}$
$\mathfrak{so}_{2k,2(l-k)+1}$	$(-1)^{(k+l(l-1)/2)\Lambda_l}$
$\mathfrak{so}_{2k,2(2p-k)}$	$(-1)^{(k+p)(\Lambda_{2p-1}+\Lambda_{2p})}$
$\mathfrak{sp}_{k,l-k}$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2\lfloor (l+1)/2\rfloor-1}}$
EVI	$(-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$

Table 1: **Original Table**

$\mathfrak{g}$	$\delta(\rho(\Lambda))$
$\mathfrak{su}_{k,2p-k}$ $(p \geq 2, 1 \leq k \leq p)$	$(-\mathbf{1})^{(\mathbf{p^2+k})\mathbf{\Lambda_p}}$
$\mathfrak{u}_l^*(\mathbb{H})$ $(l \geq 4)$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2\lfloor l/2 \rfloor -1}}$
$\mathfrak{sl}_p(\mathbb{H})$ $(p \geq 3)$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2p-1}}$
$\mathfrak{so}_{2k-1,2(l-k)+1}$ $(l \geq 2, 1 \leq k \leq \lfloor l/2 \rfloor)$	$(-\mathbf{1})^{(\mathbf{k+l(l-1)/2})(\mathbf{\Lambda_{l-1}+\Lambda_l})}$
$\mathfrak{so}_{2k,2(l-k)+1}$ $(l \geq 3, 2 \leq k \leq l)$	$(-\mathbf{1})^{(\mathbf{k+l(l+1)/2})\mathbf{\Lambda_l}}$
$\mathfrak{so}_{2k,2(2p-k)}$ $(p \geq 2, 2 \leq k \leq p)$	$(-1)^{(k+p)(\Lambda_{2p-1}+\Lambda_{2p})}$
$\mathfrak{sp}_{k,l-k}$ $(l \geq 2, 1 \leq k \leq \lfloor l/2 \rfloor)$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2\lfloor (l+1)/2 \rfloor -1}}$
EVI	$(-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$

Table 2: **Corrected Table**

Bold font marks the corrections. Some bounds on the index of the real forms are also added to the table, for without them there are apparent contradictions. For example the indicators of the same representations of $\mathfrak{so}_{3,5}$ and $\mathfrak{so}_{5,3}$ would be different.

The notation $\mathfrak{g}$ here denotes a real form of a simple complex Lie algebra. $\rho(\Lambda)$ is an irreducible complex representation of $\mathfrak{g}$ with highest weight Λ such that $\overline{\rho} \cong \rho$. A well known fact is that Λ can be written as a linear combination of fundamental

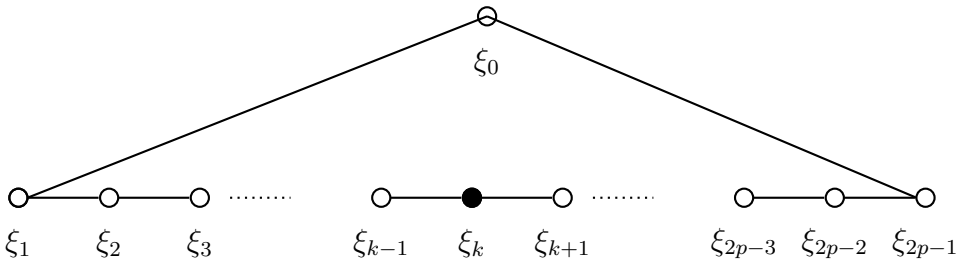
representations with coefficient of the i^{th} fundamental representation Λ_i.

Now we give the detailed calculation for the corrected terms and an unequal rank case. They serve as a sample, if the reader wants to verify the rest, the same method applies.

B.1 $\mathfrak{su}_{k,2p-k}$

We will list some structural facts about this Lie algebra and then apply the formula in Theorem 4.4.2 or Theorem 4.5.5 to compute the δ-indicator.

- Complexification: $\mathfrak{sl}_{2p}(\mathbb{C})$
- Type: A_{2p-1}
- Kac diagram:



$(p \geq 2, 1 \leq k \leq p)$

- Type of maximal compact: $A_{k-1} \oplus A_{2p-1-k} \oplus \mathbb{C}$
- $\mathfrak{su}_{k,2p-k}$ is equal rank
- Conjugate of fundamental representations: $\overline{\rho_i} = \rho_{2p-i}$
- Self-conjugate fundamental representations: ρ_p

- Strong real form: $x = \exp(\pi i \gamma_k^\vee)$

- Fundamental weights:

$$\gamma_k = \gamma_k^\vee = (\epsilon_1 + \cdots \epsilon_k) - \frac{k}{2p} \sum_{j=1}^{2p} \epsilon_j$$

- Half sum of positive co-roots:

$$2\rho^\vee = (2p-1)\epsilon_1 + (2p-3)\epsilon_2 + (2p-5)\epsilon_3 + \cdots - (2p-3)\epsilon_{2p-1} - (2p-1)\epsilon_{2p}$$

- Compute $\delta(\rho_p)$:

$$\delta(\rho_p) = \chi_{\rho_p}(x^2)\varepsilon(\rho_p) = (-1)^{\gamma_p \cdot (2\gamma_k^\vee + 2\rho^\vee)}$$

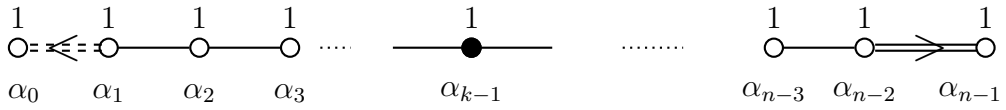
$$\begin{aligned}
\gamma_p \cdot 2\gamma_k^\vee &= (\epsilon_1 + \cdots \epsilon_p - \frac{1}{2}\sum_{j=1}^{2p} \epsilon_j) \cdot 2(\epsilon_1 + \cdots + \epsilon_k - \frac{k}{2p}\sum_{j=1}^{2p} \epsilon_j) \\
&= 2k - k - k + \frac{k}{2p} \cdot 2p = k \\
\gamma_p \cdot 2\rho^\vee &= (\epsilon_1 + \cdots \epsilon_p - \frac{1}{2}\sum_{j=1}^{2p} \epsilon_j) \cdot ((2p-1)\epsilon_1 + \cdots - (2p-1)\epsilon_{2p}) \qquad (4) \\
&= p(2p-1) - 2\sum_{j=1}^{p-1} j - \frac{1}{2}(2p)(2p-1) + \sum_{j=1}^{2p-1} j \\
&= p(2p-1) - p(p-1) = p^2
\end{aligned}$$

$$\delta(\rho_p) = (-1)^{k+p^2}$$

- Conclusion: $\delta(\rho(\Lambda)) = (-1)^{(k+p^2)\Lambda_p}$

B.2 $\mathfrak{so}_{2k-1,2(l-k)+1}$

- Complexification: $\mathfrak{so}_{2l}(\mathbb{C})$
- Type: D_l
- Kac diagram



$$(l \geq 3, 0 \leq k \leq \lfloor \frac{l-1}{2} \rfloor)$$

- Type of maximal compact: $B_{k-1} \oplus B_{l-k}$
- $\mathfrak{so}_{2k-1,2(l-k)+1}$ is NOT equal rank
- Conjugates of fundamental representations: $\overline{\rho_i} = \rho_i$ for $1 \leq i \leq l-2$ and $\overline{\rho_{l-1}} = \rho_l$ if $l \equiv 0(2)$, ρ_{l-1} and ρ_l are self-conjugate if $l \equiv 1(2)$
- Dual of fundamental representations: $\rho_i^\vee = \rho_i$ for $1 \leq i \leq l-2$ and $\rho_{l-1}^\vee = \rho_l$ if $l \equiv 1(2)$, ρ_{l-1} and ρ_l are self-dual if $l \equiv 0(2)$.
- Self-conjugate fundamental representations: ρ_i for $1 \leq i \leq l-2$, ρ_{l-1} and ρ_l if $l \equiv 1(2)$
- Hermitian fundamental representations: ρ_i for $1 \leq i \leq l-2$
- Formulas for the δ-indicators:

For Hermitian representation ρ_i:

$$\delta(\rho_i) = (-1)^{\gamma_i(2\gamma_k^\vee + 2\rho^\vee)}$$

For non-Hermitian representation ρ_i:

$$\delta(\rho_i) = \chi_{\rho_i}(x^2)\varepsilon(\widetilde{\rho_i})$$

See Theorem 4.5.5. According to [9], $\varepsilon(\widetilde{\rho_i}) = \chi_{\rho_i}(C^2)$, where C is the strong real form associated to the split real form. We can look up the table in [1] for the split form. In this case

$$C = \exp(\pi i \gamma_{(l-1)/2}^\vee)\delta_f \qquad C^2 = \exp(2\pi i \gamma_{(l-1)/2}^\vee)$$

Therefore

$$\delta(\rho_i) = (-1)^{\gamma_i(2\gamma_k^\vee + 2\gamma_{(l-1)/2}^\vee)}$$

- Strong real form: $x = \exp(\pi i \gamma_k^\vee)\delta_f$ See Theorem A.5.4

- Fundamental (co)weights:

$$\gamma_k = \gamma_k^\vee = \epsilon_1 + \cdots \epsilon_i \quad 1 \le i \le l-2 \qquad \gamma_{l-1} = \gamma_{l-1}^\vee = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{l-1} - \epsilon_l)$$

$$\gamma_l = \gamma_l^\vee = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_l)$$

- Half sum of positive co-roots:

$$2\rho^\vee = 2(l-1)\epsilon_1 + 2(l-2)\epsilon_2 + \cdots + 2\epsilon_{l-1}$$

- Compute $\delta(\rho_i)$:

For $1 \leq i \leq l-2$: $\delta(\rho_i) = (-1)^{\gamma_i(2\gamma_k^\vee + 2\rho^\vee)}$

$$\gamma_i(2\gamma_k^\vee) = 2\min(i,k)$$

$$\gamma_i(2\rho^\vee) = (\epsilon_1+\cdots+\epsilon_i)\cdot(2(l-1)\epsilon_1+\cdots+2\epsilon_{l-1}) = 2(l-1+\cdots+l-i) = 2li-(i+1)i$$

$$\delta(\rho_i) = 1 \qquad\qquad 1 \leq i \leq l-2$$

For $i = l-1$ or $i = l$ and $l \equiv 1(2)$, $\delta(\rho_i) = (-1)^{\gamma_i(2\gamma_k^\vee + 2\gamma_{(l-1)/2}^\vee)}$

$$\gamma_i(2\gamma_k^\vee) = k$$

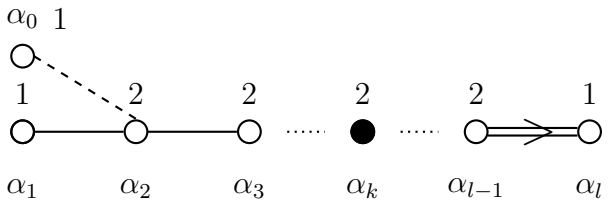
$$\gamma_i(2\gamma_{(l-1)/2}^\vee) = \frac{1}{2}(\epsilon_1 + \cdots \epsilon_{l-1} - \epsilon_l)\cdot(2(l-1)\epsilon_1 + \cdots + 2\epsilon_{l-1})$$
$$= l-1+l-2+\cdots+1 = \frac{l(l-1)}{2}$$
$$\delta(\rho_{l-1}) = \delta(\rho_l) = (-1)^{k+\frac{l(l-1)}{2}}$$

- Conclusion:

$$\delta(\rho(\Lambda)) = (-1)^{(k+\frac{l(l-1)}{2})(\Lambda_{l-1}+\Lambda_l)}$$

B.3 $\mathfrak{so}_{2k,2(l-k)+1}$

- Complexification: $\mathfrak{so}_{2l+1}(\mathbb{C})$
- Type: B_l
- Kac diagram:



$$(l \geq 3, 2 \leq k \leq l-1)$$

- Type of maximal compact: $D_k \oplus B_{l-k}$

- $\mathfrak{so}_{2k,2(l-k)+1}$ is equal rank

- Conjugates of fundamental representations: $\overline{\rho_i} = \rho_i$

- Self-conjugate fundamental representations: $\rho_i, \forall 1 \leq i \leq l$

- Strong real form: $x = \exp(\pi i \gamma_k^\vee)$

- Fundamental co-weights:

$$\gamma_k^\vee = \epsilon_1 + \cdots \epsilon_k \;\; 1 \leq k \leq l$$

- Fundamental weights:

$$\gamma_k = \epsilon_1 + \cdots \epsilon_i \;\; 1 \leq i < l \qquad\qquad \gamma_l = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_l)$$

- Half sum of positive co-roots:

$$2\rho^\vee = 2l\epsilon_1 + (2l-2)\epsilon_2 + \cdots + 4\epsilon_{l-1} + 2\epsilon_l$$

- Compute $\delta(\rho_i)$:

$$\delta(\rho_i) = \chi_{\rho_i}(x^2)\varepsilon(\rho_i) = (-1)^{\rho_i(2\gamma_k^\vee + 2\rho^\vee)}$$

For $1 \leq i < l$

$$\rho_i(2\gamma_k^\vee) = (\epsilon_1 + \cdots \epsilon_i) \cdot 2(\epsilon_i + \cdots \epsilon_k) = 2\min(i,k)$$

$\rho_i(2\rho^\vee) = (\epsilon_1 + \cdots \epsilon_i)(2l\epsilon_1 + 2(l-1)\epsilon_2 + \cdots + 2\epsilon_l)$

$$= 2(l + l - 1 + l - 2 + \cdots l - (i-1)) = 2(il - \frac{i(i-1)}{2}) = 2il - i^2 + i$$

$$\delta(\rho_i) = (-1)^{2\min(i,k)+2il-i(i-1)} = 1$$

For $i = l$

$$\rho_l(2\gamma_k^\vee) = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_l) \cdot 2(\epsilon_1 + \cdots + \epsilon_k) = k$$

$$\begin{aligned}\rho_l(2\rho^\vee) &= \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_l) \cdot (2l\epsilon_1 + 2(l-1)\epsilon_2 + \cdots + 2\epsilon_l) \\ &= l + l - 1 + l - 2 + \cdots + 1 = \frac{(l+1)l}{2}\end{aligned}$$

$$\delta(\rho_l) = (-1)^{k+\frac{(l+1)l}{2}}$$

- Conclusion: $\delta(\rho(\Lambda)) = (-1)^{(k+l(l+1)/2)\Lambda_l}$

Part II

A Comment on the Local Langlands Correspondence of $GL(2, F)$

Chapter 0: Introduction

In this part, we are going to spell out the construction of the local Langlands correspondence of $GL(2, F)$. Although this is done in [14] and various other places, we give a treatment that is easier to understand and only requires comparatively more elementary tools.

What's new here is that by looking into the construction from a more elementary perspective, we are able to provide less complicated proofs for some facts. For example, in [14], in order to prove the fact that the Langlands correspondence is independent of ψ (Theorem 3.0.6), they invoked the local constant. Here we prove it directly from properties of the Weil representation.

Throughout this part, F will denote a non-Archimedean local field with characteristic 0. The Weil group of F will be denoted $\mathcal{W}_F$. The local Langlands conjecture says there exists a unique bijection between the set $\mathfrak{G}_2(F)$, defined to be:

$$\mathfrak{G}_2(F) := \{\text{2-dim, semi-simple Deligne representations of the Weil group } \mathcal{W}_F\}/\sim$$

and the set $\boldsymbol{\mathcal{A}}_2(F)$, defined to be:

$$\boldsymbol{\mathcal{A}}_2(F) := \{\text{irreducible smooth representations of } GL(2, F)\}/\sim$$

with certain properties:

Definition 0.0.1. [Local Langlands Conjecture] There is a unique bijection

$$\boldsymbol{\pi} : \mathfrak{G}_2(F) \to \boldsymbol{\mathcal{A}}_2(F)$$

such that

$$L(\chi(\boldsymbol{\pi}(\rho)), s) = L(\chi \otimes \rho, s) \tag{1}$$

$$\epsilon(\chi(\boldsymbol{\pi}(\rho)), s, \psi) = \epsilon(\chi \otimes \rho, s, \psi) \tag{2}$$

for all $\rho \in \mathfrak{G}_2(F)$ and all characters χ of $F^\times$ and all $\psi \in \widehat{F}$, $\psi \neq 1$.

Here $L(\pi, s)$ is the L-function and $\epsilon(\pi, s, \psi)$ is the local-constant [14], and

$$\chi\pi = \pi \otimes (\chi \circ \det)$$

We have the following decomposition:

$$\mathfrak{G}_2^0(F) = \{\text{irreducible, 2-dim, semi-simple Deligne representation of } \mathcal{W}_F\}$$

$$\mathfrak{G}_2^1(F) = \{\text{reducible, 2-dim, semi-simple Deligne representations of } \mathcal{W}_F\}$$

$$\mathfrak{G}_2(F) = \mathfrak{G}_2^0(F) \cup \mathfrak{G}_2^1(F).$$

$$\boldsymbol{\mathcal{A}}_2^0(F) = \{\text{irreducible cuspidal smooth representations of } GL(2, F)\}$$

$$\boldsymbol{\mathcal{A}}_2^1(F) = \{\text{irreducible non-cuspidal smooth representations of } GL(2, F)\}$$

$$\boldsymbol{\mathcal{A}}_2(F) = \boldsymbol{\mathcal{A}}_2^0(F) \cup \boldsymbol{\mathcal{A}}_2^1(F).$$

Proposition 0.0.1. *[14, P213] Any map $\boldsymbol{\pi}$ satisfying equation (1) and (2) must take $\mathfrak{G}_2^1(F)$ to $\boldsymbol{\mathcal{A}}_2^1(F)$ and $\mathfrak{G}_2^0(F)$ to $\boldsymbol{\mathcal{A}}_2^0(F)$.*

The map

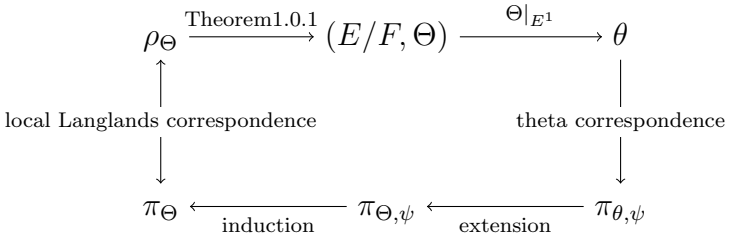
$$\boldsymbol{\pi^1} : \mathfrak{G}_2^1(F) \to \boldsymbol{\mathcal{A}}_2^1(F)$$

is clearly presented in [14, P213].

The heart of the matter is therefore to construct map:

$$\boldsymbol{\pi} : \mathfrak{G}_2^0(F) \to \boldsymbol{\mathcal{A}}_2^0(F)$$

There is more than one way to construct this map. Here, we will be using the theta correspondence. This construction is given in a more complicated manner in [14]. Here is a simplified version given by the following diagram:



The process goes clockwise beginning at $\rho_\Theta \in \mathfrak{G}_2^0(F)$.

1. $\mathfrak{G}_2^0(F) \ni \rho_\Theta \mapsto (E/F, \Theta) \in \mathbb{P}(F)$ (Definition 1.0.3), this map is described in Theorem 1.0.1.

2. $\mathbb{P}(F) \ni (E/F, \Theta) \mapsto \theta := \Theta|_{E^1}$, here E^1 is the norm 1 elements in E and $E^1 \cong SO(2, F)$. E is a tamely ramified quadratic field extension of F.

3. $\widehat{SO}(2, F) \ni \theta \mapsto \pi_{\theta,\psi} \in \widehat{SL}(2, F)$. This map is produced by (some modification of) the theta correspondence. Note that $\pi_{\theta,\psi}$ is dependent on an additive character ψ.

4. $\widehat{SL}(2,F) \ni \pi_{\theta,\psi} \mapsto \pi_\Theta \in \widehat{GL}(2,F)$. The representation π_Θ is obtained by extending $\pi_{\theta,\psi}$ and then inducing the extended representation. Note that π_Θ is independent of ψ.

The core of the construction is to use the theta correspondence for the dual pair $(O(2,F), Sp(2,F))$ in $Sp(4,F)$. This allows us to get a representation $\pi_{\theta,\psi}$ of $Sp(2,F) \cong SL(2,F)$ from θ. Note that the dual pair correspondence depends on ψ.

Since the Langlands correspondence should not involve ψ, naturally we would like to prove that the representation π_Θ is independent of ψ. Bushnell and Henniart [14] used the local constant to prove this fact. Here we derive this fact directly from properties of the Weil representation. In addition, the irreducibility of π_Θ is also naturally built in.

Chapter 1: Admissible Pairs

Most of the content in this section is from [14].

Consider a pair $(E/F, \chi)$, where E/F is a tamely ramified quadratic field extension and χ is a character of $E^\times$.

Definition 1.0.1. The pair $(E/F, \chi)$ is called *admissible* if:

1. χ does not factor through the norm map $N_{E/F} : E^\times \to F^\times$ and,

2. if $\chi|U_E^1$ does factor through $N_{E/F}$, then E/F is unramified.

Here $U_E^1 = 1 + \mathfrak{p}$ where $\mathfrak{p}$ is the maximal ideal of the discrete valuation ring in E.

Definition 1.0.2. [14, P127] Admissible pairs $(E/F, \chi)$, $(E'/F, \chi')$ are said to be *F-isomorphic* if there is an F-isomorphism $j : E \to E'$ such that $\chi = \chi' \circ j$. In the case $E = E'$, this amounts to $\chi' = \chi^\sigma$, $\sigma \in Gal(E/F)$.

Definition 1.0.3. Define $\mathbb{P}(F)$ to be the F-isomorphism classes of admissible pairs $(E/F, \chi)$.

Theorem 1.0.1. *[14] There is a bijection* $\mathbb{P}(F) \leftrightarrow \mathfrak{G}_2^0(F)$.

Here's how the map is constructed.

If $(E/F, \chi)$ is an admissible pair, then χ can be viewed as a character of the Weil group $\mathcal{W}_E$ by composing with the Artin map: $\boldsymbol{a}_E : \mathcal{W}(E) \to E^\times$. We know that $\mathcal{W}(E)$ is an index two subgroup in $\mathcal{W}(F)$, we can induce χ:

$$\rho_\chi = \mathrm{Ind}_{\mathcal{W}(E)}^{\mathcal{W}(F)} \chi$$

This induced representation is irreducible by Clifford theory, since χ not factoring through $N_{E/F}$ is equivalent to the fact that $\chi \neq \chi^\sigma$, and the Artin map is $\mathrm{Gal}(E/F)$-equivalent.

The proof of the bijectivity of this map is given in [14].

Remark 17. The admissible pairs $\mathbb{P}(F)$ also parametrize $\boldsymbol{\mathcal{A}}_2^0(F)$, and it can therefore give a bijection between $\mathfrak{G}_2^0(F)$ and $\boldsymbol{\mathcal{A}}_2^0(F)$, but it is well known that this map does not satisfy equation (1) and (2).

There are various ways to define the correct correspondence. [14] does this in Chapter 9. But it is not the optimal perspective in terms of understanding

the construction. We will explain the construction in a more understandable way. Starting in the next section, we will introduce the basics of the theory of theta correspondence.

Chapter 2: The Theta Correspondence

We are going to explain the theory of theta correspondence in the generality that suits the purpose of this thesis. First of all, we define a basic ingredient of the theta correspondence: a reductive dual pair (G, G') in $Sp(W)$. Then we will present the Schrödinger Model of the Weil representation of the Metaplectic group $Mp(W)$. The restriction of the Weil representation to the direct product of the lifts $\widetilde{G} \times \widetilde{G'}$ gives the theta correspondence between representations of $\widetilde{G}$ and $\widetilde{G'}$. We apply this to $\widetilde{O}(E) \times \widetilde{Sp}(2, F)$.

Most of the material and detailed proofs in this section can be found in [15], [16] and [17].

2.1 Reductive Dual Pair

Definition 2.1.1. A *Reductive dual pair* is a pair of subgroups (G, G') in the symplectic group $Sp(W)$ such that:

1. G is the centralizer of G' in $Sp(W)$, and G' is the centralizer of G in $Sp(W)$.

2. the actions of G and G' are completely reducible on W, i.e., any G-invariant

subspace of W has a G-invariant complement; similarly for G'.

What we will be using is a specific kind of dual pair. Let V and W be finite dimensional vector spaces over F; V is equipped with a non-degenerate symmetric bilinear form $(,)$; and W is equipped with a non-degenerate skew-symmetric bilinear form $\langle,\rangle$. Necessarily $\dim(W)$ has to be even.

Define $\mathbb{W} = V \otimes W$ with symplectic form $\langle\langle,\rangle\rangle = (,) \otimes \langle,\rangle$. This makes the pair $(O(V), Sp(W))$ a reductive dual pair in $Sp(\mathbb{W})$.

2.2 Weil Representation and the Schrödinger Model

Throughout this section W is a symplectic vector space with non-degenerate symplectic bilinear form $\langle,\rangle$ represented by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Fix X and Y two transversal Lagrangian subspaces of W.

Definition 2.2.1. The *Heisenberg group* $H(W)$ is a non-trivial extension of W by F. It is defined to be the group of pairs

$$\{(w,t) \mid w \in W, t \in F\}$$

with multiplication law:

$$(w_1,t_1)\cdot(w_2,t_2) = (w_1+w_2, t_1+t_2+\frac{1}{2}\langle w_1,w_2\rangle)$$

Thus $H(W)$ fits in the exact sequence

$$0 \to Z \to H(W) \to W \to 0$$

where $Z = \{(0,t) \mid t \in F\} \cong F$ is the center of $H(W)$.

The irreducible unitary representations of $H(W)$ are classified by their central characters.

Theorem 2.2.1 (Stone-von Neumann)**.** *The Heisenberg group $H(W)$ has a unique (up to unitary equivalence) irreducible unitary representation ρ_ψ with central character ψ.*

The proof is given in [18].

These representations can be realized on the Schwartz space $\mathcal{S} = \mathcal{S}(X)$, the space of locally constant compactly supported complex valued functions on X. We will now give a construction of $(\rho_\psi, \mathcal{S})$. Here we follow [15] very closely.

Let $H(Y) \cong Y \oplus Z$ be the abelian subgroup of the Heisenberg group associated to Y in W. Thus $H(Y)$ is a maximal abelian subgroup of $H(W)$. The character ψ of Z has a unique extension ψ_Y to $H(Y)$, given by $\psi_Y(y,t) = \psi(t)$. Let

$$S_Y = \mathrm{Ind}_{H(Y)}^{H(W)} \psi_Y$$

be the representation of $H(W)$ obtained from the character ψ_Y by smooth induction. By definition:

$$S_Y = \{f : H(W) \to \mathbb{C} \mid f(h_1 h) = \psi_Y(h_1) \cdot f(h), \forall h_1 \in H(Y), h \in H(W)$$

and there is an open subgroup $L \subset W$ such that $f(h(u,0)) = f(h)$ for all $u \in L\}$.

Then $H(W)$ acts on S_Y by right translation, in particular, Z acts by the following:

$$(0,t) \cdot f(h) = f(h \cdot (0,t)) = f((0,t) \cdot h) = \psi_Y((0,t)) f(h) = \psi(t) f(h)$$

Thus the central character is ψ. Denote this action of $H(W)$ on S_Y as ρ_ψ.

Since $W = X + Y$ is a complete polarization of W, we have the following isomorphism:

$$S_Y \cong \mathcal{S}$$

given by $f \mapsto \varphi$, $\varphi(x) = f((x, 0))$.

Lemma 2.2.2. *The resulting action of $H(W)$ on $\mathcal{S}$ is given by:*

$$\rho_\psi((x + y, t)) \cdot \varphi(x_0) = \psi(t + \langle x_0, y\rangle + \frac{1}{2}\langle x, y\rangle) \cdot \varphi(x_0 + x)$$

Now, we will explain how the representations of $H(W)$ give rise to a projective representation of $Sp(W) = \{g \in GL(W) \mid \langle w_1 g, w_2 g\rangle = \langle w_1, w_2\rangle\}$.

The group $Sp(W)$ acts on $H(W)$ naturally:

$$g \cdot (w, t) = (wg, t), \quad g \in Sp(W)$$

therefore acts on the representation ρ_ψ:

$$g \cdot \rho_\psi = \rho_\psi^g, \quad \text{where } \rho_\psi^g((w, t)) = \rho_\psi(g \cdot (w, t)) = \rho_\psi(wg, t)$$

It is easy to check that ρ_ψ^g still has central character ψ. By Theorem 2.2.1, we know $\rho_\psi \cong \rho_\psi^g$, therefore g gives rise to $T(g) : \mathcal{S} \to \mathcal{S}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{T(g)} & \mathcal{S} \\ {\scriptstyle \rho_\psi}\downarrow & & \downarrow{\scriptstyle \rho_\psi^g} \\ \mathcal{S} & \xrightarrow{T(g)} & \mathcal{S} \end{array}$$

The intertwining operator $T(g)$ is unique up to scalar in $\mathbb{C}^\times$, so the map

$$g \mapsto T(g)$$

defines a projective representation of $Sp(W)$ on $\mathcal{S}$, i.e., a homomorphism

$$Sp(W) \to GL(\mathcal{S})/\mathbb{C}^\times$$

We also present a realization of this projective representation based on the previous realization of ρ_ψ on $\mathcal{S}$.

Consider W with the complete polarization we have fixed in the beginning of this section: $W = X + Y$; we can write $g \in Sp(W)$ as

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathrm{End}(A)$, $B \in \mathrm{Hom}(X, Y)$, $C \in \mathrm{Hom}(Y, X)$ and $D \in \mathrm{End}(Y)$.

Definition 2.2.2. [15] The *standard Segal-Shale-Weil representation* ω_ψ of $Sp(W)$ is realized on $\mathcal{S}$:

$$(\omega_\psi(g) \cdot \varphi)(x)$$
$$= \int_{\ker C \backslash Y} \psi\left(\frac{1}{2}\langle xA + yC, xB\rangle + \frac{1}{2}\langle yC, xB + yD\rangle\right) \varphi(xA + yC)\ d\mu_g(y)$$

$d\mu_g(y)$ is the unique Haar measure on $(Yg^{-1} \cap Y)\backslash Y$, hence on $(\ker C)\backslash Y$, such that $\omega_\psi(g)$ preserves the L^2 norm on $\mathcal{S}$.

Remark 18. Note that this construction depends on the character ψ of F and the symplectic basis we chose. For detailed calculations, see [19] and [15].

This realization determines a cocycle, i.e., a map $c : Sp(W) \times Sp(W) \to \mathbb{C}^*$:

$$\omega_\psi(g_1 g_2) = c(g_1, g_2)\omega_\psi(g_1)\omega_\psi(g_2)$$

satisfying the cocycle condition:

$$c(g_1, g_2)c(g_1 g_2, g_3) = c(g_1, g_2 g_3)c(g_2, g_3)$$

Theorem 2.2.3. *The cocycle c can be explicitly calculated and normalized such that it is ± 1 valued. More specifically, we can find normalizing constant $m(g)$ such that the cocycle $\tilde{c}$ associated to the projective representation $\omega'_\psi := m(g)\omega_\psi(g)$ is ± 1 valued.*

Remark 19. In order to avoid putting heavy notations in this section, we will present explicit formulas for $m(g)$ and the normalized cocycle in the Appendix.

Definition 2.2.3 (Weil representation)**.** The projective representation gives rise to an ordinary representation of the two-fold covering space of $Sp(W)$; it is called the *Weil representation.*

Definition 2.2.4 (Schrödinger Model)**.**

$$Mp(W) = \{(g, \epsilon) \mid g \in Sp(W), \epsilon = \pm 1\}$$

with group multiplication

$$(g_1, \epsilon) \cdot (g_2, \delta) = (g_1 g_2, \epsilon\delta\tilde{c}(g_1, g_2))$$

is the Metaplectic group and the representation of $Mp(W)$ on $\mathcal{S}$

$$\tilde{\omega}_\psi(g, \epsilon) = \epsilon\ \omega'_\psi$$

is called the *Schrödinger model of the Weil representation.*

One property of the Weil representation we obtain from construction is the following:

Corollary 2.2.4. *[17, 1.11]*

$$\tilde{\omega}_{\psi_t} \cong \tilde{\omega}_{\psi_s} \Leftrightarrow st \in (F^\times)^2$$

2.3 Theta Correspondence

The foundation of the theory is due to Howe.

We will start with a brief introduction to the general theory, then specialize to the case we are interested in. Here we follow [15] closely.

Let (G, G') be a dual pair in $Sp(W)$, $\widetilde{G}$ and $\widetilde{G'}$ be the inverse images of G and G' in $Mp(W)$.

Lemma 2.3.1. *[18, Lemma 2.5] If two elements in $Sp(W)$ commute then their arbitrary lifts in $Mp(W)$ also commute.*

Therefore, we have a natural homomorphism:

$$j : \widetilde{G} \times \widetilde{G'} \to Mp(W)$$

hence we can consider the pull back of the Weil representation to $\widetilde{G} \times \widetilde{G'}$, namely $\tilde{\omega}_\psi \circ j$; by abuse of notation we denote this representation $\tilde{\omega}_\psi$.

The basic idea of the theta correspondence is that the Weil representation of $Mp(W)$ is very "small"; its restriction to $\widetilde{G} \times \widetilde{G'}$ should decompose into irreducibles in a reasonable way.

Suppose π is an irreducible admissible representation of $\widetilde{G}$. Let $S(\pi)$ be the maximal quotient of $\mathcal{S}$ on which $\widetilde{G}$ acts as a multiple of π. By [18, Chapter 2], there is a smooth representation $\Theta_\psi(\pi)$ of $\widetilde{G'}$ such that

$$S(\pi) \cong \pi \otimes \Theta_\psi(\pi)$$

and $\Theta_\psi(\pi)$ is unique up to isomorphism.

Theorem 2.3.2 (Howe Duality Principle)**.** *For any irreducible admissible representation π of $\widetilde{G}$*

1. *Either $\Theta_\psi(\pi) = 0$ or $\Theta_\psi(\pi)$ is an admissible representation of $\widetilde{G'}$ of finite length.*

2. *If $\Theta_\psi(\pi) \neq 0$, there is a unique $\widetilde{G'}$ invariant submodule $\Theta^0_\psi(\pi)$ such that the quotient*

$$\theta_\psi(\pi) := \Theta_\psi(\pi)/\Theta^0_\psi(\pi)$$

   is irreducible.

3. *If $\theta_\psi(\pi_1)$ and $\theta_\psi(\pi_2)$ are nonzero and isomorphic, then $\pi_1 \cong \pi_2$.*

Definition 2.3.1 (Theta Correspondence)**.** Let

$$Howe_\psi(\widetilde{G}, \widetilde{G'}) = \{\pi \in \mathrm{Irr}(\widetilde{G}) \mid \theta_\psi(\pi) \neq 0\}$$

be the set of (up to isomorphism) irreducible admissible representations of $\widetilde{G}$. The map:

$$\pi \mapsto \theta_\psi(\pi)$$

defines a bijection:

$$Howe_\psi(\widetilde{G}, \widetilde{G'}) \xrightarrow{\sim} Howe_\psi(\widetilde{G'}, \widetilde{G})$$

this bijection is referred to as the *local theta correspondence.*

We now specialize to the case we are interested in:

Let $E = F(\delta)$ be a tamely ramified quadratic field extension of F, $\delta = \sqrt{\Delta}$, $\Delta \in F^\times/(F^\times)^2$. The field E can be viewed as a 2-dimensional vector space

over F with symmetric bilinear form induced by the norm map:

$$N(\alpha_1 + \alpha_2\delta) = \alpha_1^2 - \Delta\alpha_2^2$$

with the basis: $\{1, \delta\}$, the form is represented by $\begin{pmatrix} 1 & 0 \\ 0 & -\Delta \end{pmatrix}$.

Let W be a 2-dimensional symplectic vector space over F with basis w_1, w_2 and skew-symmetric form (after choosing basis) represented by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Consider $\mathbb{W} = W \otimes V$ with basis $\{x_1 := w_1 \otimes 1, x_2 := w_1 \otimes -\frac{1}{\Delta}\delta, y_1 := w_2 \otimes 1, y_2 := w_2 \otimes \delta\}$; the skew-symmetric bilinear form is represented by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let $E^1 = \{\alpha \in E \mid N(\alpha) = 1\}$. E^1 acts on E by multiplication on the left; it's easy to check that $E^1 \cong SO(E)$ by map:

$$\alpha_1 + \alpha_2\delta \mapsto \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2\Delta & \alpha_1 \end{bmatrix}$$

and note that $Sp(2) \cong SL(2, F)$. By the theta correspondence and the Schrödinger model of Weil representation, we have the following result:

Theorem 2.3.3. *Suppose θ is a regular character of E^1. The corresponding irreducible representation $\pi_{\theta,\psi}$ of $SL(2, F)$ on the space $\mathcal{S}(E)_\theta = \{\varphi \in \mathcal{S}(E) \mid \varphi(\alpha \cdot \mu) =$*

$\theta(\mu)\varphi(\alpha),\ \forall \mu \in E^1,\ \forall \alpha \in E\}$ *is described as follows:*

$$\pi_{\theta,\psi} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \cdot \varphi(\alpha) = (a, \Delta) \cdot |a| \cdot \varphi(a\alpha)$$

$$\pi_{\theta,\psi} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \cdot \varphi(\alpha) = \psi(b \cdot N(\alpha)) \cdot \varphi(\alpha)$$

$$\pi_{\theta,\psi} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \varphi(\alpha) = \gamma(\Delta, \psi)\hat{\varphi}(\alpha)$$

where $\hat{\varphi} = \int_E \psi_E(z\alpha^\sigma)\varphi(z)\ dz$ *is the* σ*-twisted Fourier transform. See the Appendix for definitions of* γ *and* (a, Δ).

Proof. It is well known that the Metaplectic cover $Mp(4)$ splits over $SO(2)$. It also splits over $SL(2, F)$ because the orthogonal vector space V has even dimension. Let's first describe the explicit embedding of $SO(2) \cong E^1$ in $\widetilde{O}(E)$ and $SL(2, F)$ in $\widetilde{SL}(2, F)$. Define:

$$\iota : O(E) \times Sp(2) \to Sp(4)$$

$$\begin{bmatrix} t & u \\ r & s \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} at & -\Delta au & bt & bu \\ -\frac{ar}{\Delta} & as & -\frac{br}{\Delta} & -\frac{bs}{\Delta} \\ ct & -\Delta cu & dt & du \\ cr & -\Delta cs & dr & ds \end{bmatrix}$$

For the convenience of future reference, we also write down the embeddings:

$$\iota : O(E) \times 1 \to Sp(4)$$

$$\begin{bmatrix} t & u \\ r & s \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} t & -\Delta u & 0 & 0 \\ -\frac{r}{\Delta} & s & 0 & 0 \\ 0 & 0 & t & u \\ 0 & 0 & r & s \end{bmatrix}$$

$$\iota : 1 \times Sp(2) \to Sp(4)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -\frac{b}{\Delta} \\ c & 0 & d & 0 \\ 0 & -\Delta c & 0 & d \end{bmatrix}$$

The embedding $E^1 \hookrightarrow \widetilde{O}(E)$ is:

$$\alpha_1 + \alpha_2\delta \mapsto \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2\Delta & \alpha_1 \end{bmatrix} \mapsto \left(\iota(\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2\Delta & \alpha_1 \end{bmatrix}), 1 \right)$$

It is a straightforward calculation that this is a group homomorphism given the formulas in [19].

To obtain the embedding $SL(2,F) \hookrightarrow \widetilde{SL}(2,F)$, we look at [20]:

$$c(\iota(g_1), \iota(g_2)) = \beta_V(g_1 g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1}$$

where

$$\beta_V(g) = \gamma(x(g), \frac{1}{2}\psi))^{-m}(x(g), \det(V))\gamma(\frac{1}{2}\psi \circ V)^{-j}$$

$m = \dim V$, g is in the j-th cell $P\tau_j P$.

The embedding $s : SL(2,F) \hookrightarrow \widetilde{SL}(2,F)$ is defined as follows:

$$s(g) = (\iota(g), m(\iota(g))^{-1}\beta_V(g))$$

It is easily verified that this is a group homomorphism. One important thing to verify is that $m(\iota(g))^{-1}\beta_V(g)$ takes ± 1 value. Further calculation shows:

$$s: \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mapsto \left(\iota(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}), (a, \Delta) \right)$$

$$s: \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mapsto \left(\iota(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}), (-1, \Delta) \right)$$

generate this embedding.

It is an elementary consequence of the theory of the theta correspondence that the restriction $\widetilde{\omega}_\psi$ to $SO(2) \times SL(2, F)$ can be written as direct sum:

$$\sum_i \theta_i \boxtimes \pi_i$$

where θ_i is a character of $SO(2)$, and π_i is an irreducible representation of $SL(2, F)$.

Let's first determine the irreducible subspace on which $SO(2)$ acts with the character θ, i.e., the θ-isotypic:

$$\mathcal{S}(E)_\theta = \{\varphi \in \mathcal{S}(E) \mid \varphi(\alpha \cdot \mu) = \theta(\mu)\varphi(\alpha),\ \forall \mu \in E^1,\ \forall \alpha \in E\}$$

The way $SL(2, F)$ acts on $\mathcal{S}(E)_\theta$ is completely determined by the embedding s and the Weil representation $\widetilde{\omega}_\psi$:

$$\pi_{\theta,\psi} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \varphi(\alpha) = \widetilde{\omega}_\psi \left(s(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) \right)$$

Formulas for the Schrödinger model of $\widetilde{\omega}_\psi$ is given in the Appendix. The Theorem then follows by simple calculation.

□

Remark 20. Notice that this correspondence is 2-to-1, because θ and θ^σ both corresponds to isomorphic $SL(2,F)$ representations. Fortunately, this won't get in the way of constructing the Langlands correspondence, since later we will be using the full information of Θ to construct a representation of $GL(2,F)$. Θ and Θ^σ will correspond to different representations of $GL(2,F)$.

Chapter 3: The Langlands Correspondence

Let $(E/F, \Theta)$ be an admissible pair. We have constructed an irreducible representation $\pi_{\theta,\psi}$ of $SL(2,F)$; now we would like to inflate this representation to a representation of $GL(2,F)$ by inflating step by step through the following inclusions:

$$SL(2,F) \subset GL(2,F)_\circ \subset GL(2,F)_N \subset GL(2,F).$$

Definition 3.0.1. Define

$$GL(2,F)_\circ = SL(2,F)Z$$

where

$$Z = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \mid x \in F^\times \right\}.$$

In other words, $GL(2,F)_\circ = \{g \in GL(2,F) \mid \det(g) \in (F^\times)^2\}$

To extend $\pi_{\theta,\psi}$, we need to know a little more about the Langlands correspondence.

Proposition 3.0.1. *[14, Proposition 33.4] If $\boldsymbol{\pi}$ is the Langlands correspondence in Definition 0.0.1, then for $\rho \in \mathfrak{G}_2^0(F)$ and $\pi = \boldsymbol{\pi}(\rho)$, $\chi_\pi = \det \rho$. Here χ_π is the central character of π.*

Lemma 3.0.2.

$$\det \rho_\Theta = \kappa \otimes \Theta|_F$$

where $\kappa(x) = (x, \Delta)$, $\rho_\Theta = Ind_{\mathcal{W}_E}^{\mathcal{W}_F}\Theta$

These conditions impose some constraint on how we may extend the representation to $GL(2,F)_\circ$. We are required to define:

Definition 3.0.2.

$$\pi(\Theta,\psi)_\circ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \cdot \varphi(\alpha) = \kappa(x)\Theta(x)\varphi(\alpha)$$

is an irreducible representation of $GL(2,F)_\circ$ on $\mathcal{S}(E)_\theta$.

Next we pass from $GL(2,F)_\circ$ to $GL(2,F)_N$.

Definition 3.0.3.

$$GL(2,F)_N = \{g \in GL(2,F) \mid \det(g) \in N(E^\times)\}$$

$GL(2,F)_N$ is generated by the group $GL(2,F)_\circ$ and $N(E^\times) \cong \{ \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} \mid \gamma \in N(E^\times)\}$. Let's extend the representation to $\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}$. Let $\pi_{\Theta,\psi}$ denote the

representation of $GL(2,F)_N$ extended from $\pi(\Theta,\psi)_\circ$. For consistency, $\pi_{\Theta,\psi}$ has to satisfy:

$$\pi_{\Theta,\psi}\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}\cdot\varphi(\alpha)=\pi(\Theta,\psi)_\circ\begin{bmatrix}\gamma^2 & 0\\ 0 & 1\end{bmatrix}\cdot\varphi(\alpha)$$
$$=\Theta(\zeta^2)\cdot|\gamma|\cdot\varphi(\sigma(\zeta^2)\alpha)$$

Here $\zeta\in E$ is defined by $N(\zeta)=\gamma$.

Definition 3.0.4.

$$\pi_{\Theta,\psi}\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}\cdot\varphi(\alpha)=\Theta(\zeta)\cdot|\gamma|^{\frac{1}{2}}\cdot\varphi(\sigma(\zeta)\alpha),\quad\forall\gamma\in N(E^\times)$$

$$\pi_{\Theta,\psi}\begin{bmatrix}x & 0\\ 0 & x\end{bmatrix}\cdot\varphi(\alpha)=(x,\Delta)\Theta(x)\varphi(\alpha),\quad\forall x\in F^\times$$

together with the action of $SL(2,F)$ via $\pi_{\theta,\psi}$ defines a representation of $GL(2,F)_N$.

Remark 21. This extension is independent of the choice of ζ since if we choose another element with norm α, it is going to be of the form $\zeta\mu$, $\mu\in E^1$.

$$\Theta(\zeta\mu)|\gamma|^{\frac{1}{2}}\varphi(\sigma(\zeta)\sigma(\mu)\alpha)=\Theta(\zeta)\theta(\mu)|\gamma|^{\frac{1}{2}}\theta(\sigma(\mu))\varphi(\sigma(\zeta)\alpha)$$
$$=\Theta(\zeta)\theta(N(\mu))|\gamma|^{\frac{1}{2}}\varphi(\sigma(\zeta)\alpha)=\Theta(\zeta)|\gamma|^{\frac{1}{2}}\varphi(\sigma(\zeta)\alpha)$$

Finally, we would like to obtain an irreducible representation of $GL(2,F)$ from $\pi_{\Theta,\psi}$. Since $GL(2,F)_N$ is an index 2 subgroup of $GL(2,F)$, we would like to induce $\pi_{\Theta,\psi}$. Note that the non-trivial coset of $GL(2,F)_N$ in $GL(2,F)$ is represented by matrix $\begin{bmatrix}\eta & 0\\ 0 & 1\end{bmatrix}$, where $\eta\in F^\times\backslash N(E^\times)$.

Lemma 3.0.3. *For $t \in F^\times$, $\pi^t_{\theta,\psi} \cong \pi_{\theta,\psi_t}$.*

Here $\pi^t_{\theta,\psi}$ is defined as

$$\pi^t_{\theta,\psi}(A) = \pi_{\theta,\psi}(TAT^{-1})$$

where $T = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ and $\psi_t(\alpha) = \psi(t\alpha)$. This lemma is easy to prove from the formulas.

Lemma 3.0.4.

$$\pi_{\theta,\psi} \text{ is } \begin{cases} \cong \pi_{\theta,\psi_t} & t \in N(E^\times) \\ \ncong \pi_{\theta,\psi_t} & t \notin N(E^\times) \end{cases}$$

The fact $\pi_{\theta,\psi} \cong \pi_{\theta,\psi_t}$ when $t \in N(E^\times)$ is easy to prove from the formulas by writing down an explicit intertwining operator, while the second part can be observed from the difference in their character formulas. Such character formulas are presented in [21].

Proposition 3.0.5.

$$\pi_{\Theta,\psi} \text{ is } \begin{cases} \cong \pi_{\Theta,\psi_t} & t \in N(E^\times) \\ \ncong \pi_{\Theta,\psi_t} & t \notin N(E^\times) \end{cases}$$

This is an elementary consequence of the previous lemmas.

Definition 3.0.5.

$$\pi_\Theta = \mathrm{Ind}_{GL(2,F)_N}^{GL(2,F)} \pi_{\Theta,\psi}$$

Theorem 3.0.6. *π_Θ is irreducible and independent of ψ.*

Proof. $GL(2,F)_N$ is an index 2 subgroup in $GL(2,F)$. By Clifford theory, the induced representation π_Θ is irreducible if and only if $\pi^\eta_{\Theta,\psi}$ is not isomorphic to $\pi_{\Theta,\psi}$. η is not in $N(E^\times)$ by definition, therefore Proposition 3.0.5 implies π_Θ is irreducible.

To prove the independence of ψ, we only need to consider $\mathrm{Ind}_{GL(2,F)_N}^{GL(2,F)}\pi_{\Theta,\psi}$ and $\mathrm{Ind}_{GL(2,F)_N}^{GL(2,F)}\pi_{\Theta,\psi_\eta}$. Note the representation $\mathrm{Ind}_{GL(2,F)_N}^{GL(2,F)}\pi_{\Theta,\psi}$ restricted to $GL(2,F)_N$ decomposes into $\pi_{\Theta,\psi}\oplus\pi^\eta_{\Theta,\psi}$. By Frobenius reciprocity we have:

$$\mathrm{Ind}_{GL(2,F)_N}^{GL(2,F)}\pi_{\Theta,\psi}\cong\mathrm{Ind}_{GL(2,F)_N}^{GL(2,F)}\pi_{\Theta,\psi_\eta}$$

hence the independence of ψ. □

Now we have paved most of our way towards the Langlands correspondence. We will state a partial correspondence theorem.

Definition 3.0.6. Let $\rho\in\mathcal{G}_2^0(F)$; one say that ρ is *imprimitive* if there exists a separable quadratic extension E/F and a character ξ of $E^\times$ such that $\rho\cong\mathrm{Ind}_{E/F}\xi$. Let $\mathcal{G}_2^{im}(F)$ denote the set of imprimitive equivalence classes $\rho\in\mathcal{G}_2^0(F)$.

Theorem 3.0.7. *[14, 40.1]*

$$\rho_\Theta\mapsto(E/F,\Theta)\mapsto\pi_\Theta$$

is a bijection $\mathcal{G}_2^{im}(F)\leftrightarrow\mathcal{A}_2^0(F)$ *and it satisfies Equation 2 and Equation 1.*

The proof of this theorem can be found in [14] section 39. One only need to make the observation that $\pi_{\Theta,\psi}\cong\xi_\varkappa(\Theta,\psi)$, the latter notation is from [14]. The intertwining operator is presented in [14, 39.2].

Remark 22. When the residual characteristic of F is odd, we have $\mathfrak{G}_2^{im}(F) = \mathfrak{G}_2^0(F)$. If $p = 2$, we have $\mathfrak{G}_2^{im}(F) \subsetneq \mathfrak{G}_2^0(F)$, in which case the theorem only provides a partial correspondence.

Appendix C: Formulas for Cocycles

In this section, W is a symplectic vector space with skew-symmetric bilinear form $\langle , \rangle$, X and Y are two transversal Lagrangian subspaces of W. $Sp(W)$ has Bruhat decomposition $Sp(W) = P\Omega P$ where P is the stablizer of Y, Ω is the Weyl group.

Theorem 3.0.8. *[19, Theorem4.1]*

$$c(g_1, g_2) = \textit{Weil index of } w \mapsto \psi(\frac{1}{2}\langle w, w \cdot \rho\rangle)$$

where the isometry class of ρ is given by the Leray invariant $q(Y, Yg_2^{-1}, Yg_1)$

Lemma 3.0.9. *[19, Lemma5.1] There exists unique map $f : Sp(W) \to F^\times/(F^\times)^2$ such that the following holds:*

(i) $f(p_1gp_2) = f(p_1)f(g)f(g_2) \quad \forall p_1, p_2 \in P$

(ii) $f(\tau_S) = 1$ *for all subsets* $S \subset \{1, 2, \cdots, n\}$

(iii) $f(p) = \det(p|_Y)(F^\times)^2 \quad \forall p \in P$

Moreover, such a function is uniquely defined by

$$f(p_1\tau_S p_2) = \det(p_1 p_2|_Y)(F^\times)^2$$

See [19, P345] for the definition of τ_S.

Definition 3.0.7. Define the normalizing constant

$$m(g) = \gamma_F(f(g), \frac{1}{2}\psi)^{-1}\left(\gamma_F(\frac{1}{2}\psi)\right)^{-j}$$

for $g \in \Omega_j P\tau_S P$ with $j = |S|$, $\gamma_F(\frac{1}{2}\psi)$ is the Weil index of $\alpha \mapsto \frac{1}{2}\psi(\alpha^2)$, $\gamma_F(a,\psi) = \gamma_F(\psi_a)/\gamma_F(\psi)$.

Theorem 3.0.10. *[19, P364] If* $\dim W = 2$*, then*

$$\tilde{c}(g_1 g_2) = (f(g_1), f(g_2))(-f(g_1)f(g_2), f(g_1 g_2))$$

$$f : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{cases} d(F^\times)^2 & c = 0 \\ c(F^\times)^2 & c \neq 0 \end{cases}$$

Theorem 3.0.11. *Here are the explicit formulas for the Weil representation* $\tilde{\omega}_\psi$ *of*

$Mp(4)$:

$$\tilde{\omega}_\psi\left(\begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix}, \epsilon\right)\cdot\varphi(\alpha) = \epsilon\cdot\frac{\gamma(\frac{1}{2}\psi)}{\gamma(\det A\cdot\frac{1}{2}\psi)}|\det A|^{\frac{1}{2}}\varphi(\alpha\cdot A)$$

$$\tilde{\omega}_\psi\left(\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}, \epsilon\right)\cdot\varphi(x) = \epsilon\cdot\psi\left(\frac{1}{2}\langle x, x\cdot B\rangle\right)\varphi(x)$$

$$\tilde{\omega}_\psi\left(\tau_{\{1\}}, \epsilon\right)\cdot\varphi(x) = \epsilon\cdot\gamma(\frac{1}{2}\psi)^{-1}\int_{Y/Y_2}\psi(-b_1\alpha_1)\varphi(-b_1x_1+\alpha_2x_2)d\mu_g(b_1)$$

$$\tilde{\omega}_\psi\left(\tau_{\{2\}}, \epsilon\right)\cdot\varphi(x) = \epsilon\cdot\gamma(\frac{1}{2}\psi)^{-1}\int_{Y/Y_1}\psi(-b_2\alpha_2)\varphi(\alpha_1x_1-b_2x_2)d\mu_g(b_2)$$

$$\tilde{\omega}_\psi\left(\tau_{\{1,2\}}, \epsilon\right)\cdot\varphi(x) = \epsilon\cdot\gamma(\frac{1}{2}\psi)^{-2}\int_{Y}\psi(\langle\alpha, y\rangle)\varphi(y)d\mu_g(y)$$

$$\mu_p\{0\} = |\det p|_Y|^{-1/2}$$

$Y_i = span_{\mathbb{C}}\{y_i\}$, $y\in Y$ *is represented by coordinates* (b_1, b_2) *with basis* $\{y_1, y_2\}$.

$\alpha\in E$ *is represented by coordinates* (α_1, α_2) *with basis* $\{1, -\frac{1}{\Delta}\delta\}$

Bibliography

[1] Arkadij L Onishchik and Ernest B Vinberg. *Lie groups and algebraic groups*. Springer Science & Business Media, 1990.

[2] Nagayoshi Iwahori. On real irreducible representations of Lie algebras. *Nagoya Math. J.*, 14:59–83, 1959.

[3] Jacques Tits. *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen.* Springer-Verlag, 1967.

[4] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98. Springer Science & Business Media, 2003.

[5] Dipendra Prasad and Dinakar Ramakrishnan. Self-dual representations of division algebras and Weil groups: a contrast. *American Journal of Mathematics*, 134(3):749–772, 2012.

[6] David A Vogan. *Representations of real reductive Lie groups*, volume 15. Birkhauser, 1981.

[7] Jeffrey Adams, Marc Van Leeuwen, Peter Trapa, and David A Vogan Jr. Unitary representations of real reductive groups. *arXiv preprint arXiv:1212.2192*, 2012.

[8] Nicolas Bourbaki. *Lie groups and Lie algebras: chapters 7-9*, volume 7. Springer Science & Business Media, 2008.

[9] Jeffrey Adams. The real Chevalley involution. *Compositio Mathematica*, 150(12):2127–2142, 2014.

[10] Jeffrey Adams, Dan Barbasch, and David A Vogan. *The Langlands classification and irreducible characters for real reductive groups*, volume 104. Springer Science & Business Media, 2012.

[11] Tonny Albert Springer. *Linear algebraic groups*. Springer Science & Business Media, 2010.

[12] Ian Grant Macdonald. *Affine Hecke algebras and orthogonal polynomials*, volume 157. Cambridge University Press, 2003.

[13] Jeffrey Adams. Nonlinear covers of real groups. *International Mathematics Research Notices*, 2004(75):4031–4047, 2004.

[14] Colin J Bushnell and Guy Henniart. *The local Langlands conjecture for $GL(2)$*, volume 335. Springer Science & Business Media, 2006.

[15] Stephen Kudla. Notes on the local theta correspondence. *unpublished notes, available online*, 1996.

[16] Dipendra Prasad. Weil representation, Howe duality, and the theta correspondence. *Theta functions: from the classical to the modern*, 1:105–127, 1993.

[17] Roger Howe. On a notion of rank for unitary representations of the classical groups. In *Harmonic Analysis and Group Representation*, pages 224–331. Springer, 2010.

[18] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. Correspondances de Howe sur un corps p-adique, volume 1291 de. *Lecture Notes in Mathematics*.

[19] R Ranga Rao. On some explicit formulas in the theory of Weil representation. *Pacific Journal of Mathematics*, 157(2):335–371, 1993.

[20] Stephen S Kudla. Splitting metaplectic covers of dual reductive pairs. *Israel Journal of Mathematics*, 87(1-3):361–401, 1994.

[21] PJ Sally and Joseph A Shalika. Characters of the discrete series of representations of $SL(2)$ over a local field. *Proceedings of the National Academy of Sciences*, 61(4):1231–1237, 1968.