Notes on Tamagawa Number

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1 Tamagawa Number

Definition 1 (Tamagawa Number)**.** [1, P262] The invariant volume of G_A/G_K (if it exists) obtained with respect to the canonical choice of Tamagawa measure is called the *Tamagawa number* of G, denoted $\tau(G)$.

Remark 1. For G semi-simple, G_A/G_K has finite invariant volume because the fact that G is a perfect group, i.e. $G = [G, G]$, therefore the rational character group of G is trivial, and [1, P260 Theorem 5.5]

Remark 2. It looks like from the definition $\tau(G)$ depends on the number field K, so it should be $\tau_K(G)$. If L is a finite extension of K, we have $\tau_L(G) = \tau_K(R_{L/K}(G))$. Weil showed that in fact the Tamagawa number is independent of Weil restriction, i.e., $\tau_L(G) = \tau_K(R_{L/K}(G)) = \tau_K(G)$. This is proved with details in the paper by Oesterlé "Nombres de Tamagawa et groupes unipotentes en caractéristique p". This allow us to reduce to the case $K = \mathbb{Q}$.

Definition 2 (Tamagawa Measure)**.** [1, P261] Assume G is connected. The Haar measure on G_A can be constructed using a left invariant rational differential K-form ω on G of degree $n = \dim G$. More precisely, ω induces a left invariant measure ω_ν on G_{K_ν} for each $\nu \in V^K$ as in Example3. Let's choose numbers λ_ν for $\nu \in V_f^K$ (called *convergence coefficients*) such that $\prod_\nu \lambda_\nu \omega_\nu(G_{\mathcal{O}_\nu})$ converges absolutely (for example we can set λ_ν to be $\frac{1}{\omega_\nu(G_{\mathcal{O}_\nu})}$)

Treating G_A as the restricted topological product of the G_{K_ν} with respect to the distinguished subgroups $G_{\mathcal{O}_\nu}$, we can use the construction in Section 2 to obtain a Haar measure τ on G_A, called the *Tamagawa measure* corresponding to the set of convergence coefficiens $\lambda = (\lambda_\nu)$.

Remark 3. [1, P262] It is well kown that the convergence coefficients used in the definition of Tamagawa measure can be chosen canonically; in particular, for G semisimple they are not even necessary (i.e., one can put $\lambda_\nu = 1$ for all ν).

Remark 4. [1, P261] τ is actually independent of the choice of ω.

Suppose we have another left-invariant rational differential K-form ω', it will be written as a constant times the original ω, i.e., $\omega' = c\omega$ for some $c \in K^*$, then for every $\nu \in V^K$ the corresponding measure ω'_ν on K_ν induced by ω' is exactly $\omega'_\nu = ||c||_\nu^n \omega_\nu$, where $||\ ||_\nu$ is the normalized valuation introduced in Section 5.1. Therefore the τ' associated to ω' and the same convergence coefficients as τ, then $\tau' = (\prod_\nu ||c||_\nu^n)\tau = \tau$, the second equality is by product formula in Proposition 1.

2 Results Needed from Measure Theory

Definition 3 (Resricted Topological Product)**.** [1, P161] Let $\{X_\lambda\}_{\lambda \in Lambda}$ be a family of locally compact topological spaces, indexed by a countable set of indices Λ. Assume that open compact subsets $K_\lambda \subset X_\lambda$ are fixed for almost all $\lambda \in \Lambda$. Consider the space X whose elements are the families $x = \{x_\lambda\}_{\lambda\in\Lambda}$ where $x_\lambda \in X_\lambda$, and $x_\lambda \in K_\lambda$ for almost all $\lambda \in \Lambda$. Introduce a topology on X, taking for a fundamental system of open sets to be all sets of the form $\prod U_\lambda$, where $U_\lambda \subset X_\lambda$ is open for all λ, and $U_\lambda = K_\lambda$ for almost all λ. The space X with this topology is called the *restricted topological product* of X_λ with respect to the distinguished subsets K_λ.

Some straightforward properties of this construction.

Lemma 1. *[1, P161]*

1. *For any finite subset S of Λ such that K_λ is defined for each $\lambda \in \Lambda\backslash S$, put $X_S = \prod_{\lambda\in S} X_\lambda \times \prod_{\lambda\in\Lambda\backslash S} K_\lambda$; then X_S is open in X and the topology of X induces the direct product topology on X_S.*

2. *Each X_S is locally compact and $X = \cup_S X_S$, where the union is taken over all finite subsets S of Λ such that K_λ is given for each $\lambda \in \Lambda\backslash S$; consequently X is locally compact.*

3. *If $\{G_\lambda\}_{\lambda\in\Lambda}$ is a family of locally compact topological groups and open compact subgroups K_λ of G_λ are given for almost all λ, then the restricted topological product G of G_λ with respect to the K_λ is a locally compact topological group.*

2.1 The Construction of Haar Measure on the Restricted Topological Product

By Lemma 1 (3) we know that G has a Haar measure, we will construct this Haar measure from the Haar measures μ_λ on G_λ.

It is convenient to normalize μ_λ such that $\mu_\lambda(K_\lambda) = 1$.

For any finite subset $S \subset \Lambda$ such that K_λ is given for each $\lambda \in \Lambda\backslash S$, one has μ_S on $G_S = \prod_{\lambda\in S} G_\lambda \times \prod_{\lambda\in\Lambda\backslash S} K_\lambda$, $\mu_S = \mu_1 \times \mu_2$ where μ_1 is the usual finite product of μ_λ on $\prod_{\lambda\in S} G_\lambda$ and μ_2 is the Haar measure on the compact group $K_S = \prod_{\lambda\in\Lambda\backslash S} K_\lambda$ normalized by $\mu_2(K_S) = 1$. It's clear that if $S_1 \subset S_2$ then $G_{S_1} \subset G_{S_2}$, and the μ_S are consistent on the overlaps. Therefore, using countable additivity and representing G as the countable union $G = \cup_S G_S$, we obtain the desired μ on G.

Remark 5. Sometimes, when defining μ, it will be useful to waive the condition $\mu_\lambda(K_\lambda) = 1$ and instead to require absolute convergence of $\prod \mu_\lambda(K_\lambda)$ over all λ for which K_λ is defined. Then the measure constructed above is replaced by $c\mu$, where $c = \prod \mu_\lambda(K_\lambda)$.

Explicit examples of Haar measures can be obtained by integrating differential forms.

2.2 How to Compute Explicit Left-Invariant Differential Forms for Certain Algebraic Groups

Definition 4 (Differential Form)**.** [1, P165] A *differential form of degree* n in the neighborhood of x_0 is an expression of the form $\omega = f(x)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, where f is an analytic function in the neighborhood of x_0, $X \ni x_0$ is an analytic variety over a complete field K_ν. $x_1, \cdots, x_n$ are the local coordinates for a neighborhood of x_0.

Induced map on differential form

Suppose $F : Y \to X$ is an analytic map of two n-dimensional varieties, $y_0 \in Y$ is a point satisfying $F(y_0) = x_0$, and $y_1, \cdots, y_n$ are local coordinates in a neighborhood of y_0. If F is given by

$$(y_1, \cdots, y_n) \mapsto (F(y_1, \cdots, f_n), \cdots, F(x_1, \cdots, x_n))$$

then the induced map F^* acts on the differential form ω is defined to be

$$F^*(\omega) = f(F(y))dF_1(y_1, \cdots, y_n) \wedge \cdots \wedge dF_n(y_1, \cdots, y_n)$$

ω and f are defined in Definition 4, $dF_i(y_1, \cdots, y_n) = \sum_{j=1}^{n} \frac{\partial F_i}{\partial y_j} dy_j$

Definition 5. [1, P165] We say that a differential form ω is *invariant* with respect to an analytic automorphism $F : X \to X$ if $F^*(\omega) = \omega$.

Now let X be a smooth algebraic variety defined over K instead of an analytic one

Definition 6. A K-defined *system of local parameters* in the neighborhood of x_0 of X is a system of K-rational functions $x_1, \cdots, x_n$ from X to $\mathbb{A}$ defined at x_0, such that the differential $d_{x_0}\varphi$ of the rational map $\varphi : X \to \mathbb{A}^n$ given by $\varphi : x \mapsto (x_1(x), \cdots, x_n(x))$ is an isomorphism of tangent spaces.

Definition 7. [1, P166] An n-dimensional *differential form* over K in the neighborhood of x_0 is defined as an expression of the form $\omega = f(x)dx_1 \wedge \cdots dx_n$ where f is a K-rational function on X.

Remark 6. The definition of induced map on differential forms (transformation of differential forms) under rational map is similar to the one above.

Note that if X is defined over a complete field K_ν and $x \in X_{K_\nu}$, then any rational differential K_ν-form in a neighborhood of x_0 can also be viewed as an analytic differential form on X_{K_ν} in a neighborhood of x_0.

Computing explicit left-invariant differential forms for the algebraic groups GL_n and SL_n

Example 1 ($G = GL_n$). [1, P166] For a system of local parameters, we can take $x_{ij} : G \to \mathbb{A}$ such that x_{ij} applied to a matrix $X \in G$ is the $i - j$ entry of the matrix.

Let an n-dimensional differential form be $\omega = f(X)dx_{11} \wedge \cdots \wedge dx_{nn}$. We are going to use the left-invariant property of ω to determine f. The left action is defined as:

$$\lambda_A : G \to G \qquad \lambda_A(X) = A \cdot X$$

ω being left invariant is equivalent to

$$\begin{aligned}\lambda_A^*(\omega) = \omega &\Leftrightarrow f(\lambda_A(X))dx'_{11} \wedge \cdots \wedge dx'_{nn} = f(X)dx_{11} \wedge \cdots \wedge dx_{nn} \\ &\Leftrightarrow f(AX)d(\sum_k a_{1k}x_{k1}) \wedge \cdots \wedge d(\sum_k a_{nk}x_{kn}) = f(X)dx_{11} \wedge \cdots \wedge dx_{nn}\end{aligned}$$

$$d(\sum_k a_{ik}x_{kj}) = \sum_k a_{ik}dx_{kj}$$

therefore

$$d(\sum_k a_{1k}x_{k1}) \wedge \cdots \wedge d(\sum_k a_{nk}x_{kn}) = (\det A)^n dx_{11} \wedge \cdots \wedge dx_{nn}$$

$$\lambda_A^*(\omega) = \omega \Leftrightarrow f(AX)(\det A)^n = f(X)$$

Let $X = I$ the identity matrix, we have

$$f(A) = f(I)(\det A)^{-n}$$

and consequently, let $c = f(I)$,

$$\omega = \frac{cdx_{11} \wedge \cdots \wedge dx_{nn}}{(\det X)^n}$$

Example 2 ($G = SL_2$). [1, P166] For a system of local parameters in the neighborhood of 1 we take the functions x, y, z associated to the corresponding components of the matrix $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G$, where $t = \frac{1+yz}{x}$.

The left action is defined the same way as the previous example, and $A =$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The condition of ω being left invariant is equivalent to the following

$$\begin{aligned} f(AX)d(ax+bz)\wedge d(ay+b\frac{1+yz}{x})\wedge d(cx+dz) &= f(X)dx\wedge dy\wedge dz \\ f(AX)(a^2d+\frac{abdz}{x}-bac-\frac{b^2zc}{x})dx\wedge dy\wedge dz &= f(X)dx\wedge dy\wedge dz \\ f(AX)(a+\frac{bz}{x}) = f(AX)(\frac{ax+bz}{x}) &= f(X) \\ f(AX)(AX)_{11} &= f(X)(X)_{11} \\ f(A) &= \frac{f(I)}{x} \\ \omega &= \frac{c}{x}dx\wedge dy\wedge dz \end{aligned}$$

Now we use another system of local coordinates on $G_{\mathbb{R}} = SL_2(\mathbb{R})$. According to Iwasawa decomposition every matrix can be written uniquely as a product of three matrices:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \qquad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} (\alpha > 0), \qquad \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

Take φ, α, u as (analytic) coordinates on $G_{\mathbb{R}}$, the previous x, y, z has relations with φ, α, u as follows:

$$x = \alpha\cos\varphi, \qquad y = \alpha u\cos\varphi - \alpha^{-1}\sin\varphi, \qquad z = \alpha\sin\varphi$$

By plugging in these to the previous formula for ω we get

$$\begin{aligned} \omega = &\frac{f(I)}{\alpha\cos\varphi}(\cos\varphi d\alpha - \alpha\sin\varphi d\varphi) \\ &\wedge [(u\cos\varphi - \frac{\sin\varphi}{\alpha^2})d\alpha + (-\alpha u\sin\varphi + \alpha^{-1}\cos\varphi)d\varphi + \alpha\cos\varphi du] \wedge (\sin\varphi d\alpha + \alpha\cos\varphi d\varphi) \\ &= \frac{c}{\alpha\cos\varphi}(\alpha^2\cos^3\varphi + \alpha^2\cos\varphi\sin^2\varphi)d\varphi\wedge d\alpha\wedge du \\ &= c\alpha d\varphi\wedge d\alpha\wedge du \end{aligned}$$

2.3 Determination of Measure Corresponding to a Differential Form

Let $\omega = f(x)dx_1 \wedge \cdots dx_n$ be a nonzero local n-dimensional differential form in some neighborhood of x_0 in an n-dimensional analytic variety X. Then on this neighborhood we can define the measure $\mu = |f(x)|_\nu |dx_1|_\nu \times \cdots \times |dx_n|_\nu$, it is to say

$$\mu(E) = \int_E |f(x)|_\nu |dx_1|_\nu \cdots |dx_n|_\nu$$

where $|dx|_\nu$ is the (additive) Haar measure on K_ν, it is the ordinary Lebesgue measure if $K = \mathbb{R}$ or $\mathbb{C}$, and in the ν-adic case it is normalized so that $\mathcal{O}_\nu$ has measure 1.

If ω is defined over the entire variety X, then the local measure described above extends to a measure on the entire variety. Denote the measure as ω_ν [1, P168]

Example 3 ($G = SL_2$). [1, P168]

Let $K = \mathbb{Q}$ and ν the usual p-adic valuation. Let ω_p be the measure on $SL_2(\mathbb{Q}_p)$ corresponding to the differential form $\omega = \frac{1}{x} dx \wedge dy \wedge dz$.

To illustrate p-adic integration, we compute the volume of $\omega_p(SL_2(\mathbb{Z}_p))$ under the measure ω_p, i.e., $\omega_p(SL_2(\mathbb{Z}_p))$. To do so, note that $SL_2(\mathbb{Z}_p)$ has principle congruent subgroup $SL_2(\mathbb{Z}_p, p)$ which is the kernel of the map $SL_2(\mathbb{Z}_p) \to SL_2(F_p)$, we have short exact sequence

$$0 \to SL(\mathbb{Z}_p, p) \to SL_2(\mathbb{Z}_p) \to SL_2(F_p) \to 0$$

Since $SL_2(F_p)$ is finite, we have

$$\omega_p(SL_2(\mathbb{Z}_p)) = |SL_2(F_p)| \omega_p(SL_2(\mathbb{Z}_p, p))$$

It's easy to count the number of elements in $SL_2(F_p)$, it is $p(p^2 - 1)$. We only need to compute the other term.

$$\omega_p(SL_2(\mathbb{Z}_p, p)) = \int_{SL_2(\mathbb{Z}_p, p)} |\frac{1}{x}|_p |dx|_p |dy|_p |dz|_p$$

Note that every matrix in $SL_2(\mathbb{Z}_p, p)$ has diagonal entries congruent to 1 mod p, and off diagonal entries congruent to 0 mod p.

$|\frac{1}{x}|_p = 1$ for all $x \in SL_2(\mathbb{Z}_p, p)$ since the 1,1 element of x is congruent to 1 mod p. Note that $|dx|_\nu$ is the Haar measure on K_ν, which means $|dx|_p$ is the Haar measure μ_p on $\mathbb{Q}_P$, therefore

$$\omega_p(SL_2(\mathbb{Z}_p, p)) = |\frac{1}{x(SL_2(\mathbb{Z}_p, p))}|_p \cdot \mu_p(x(SL_2(\mathbb{Z}_p, p))) \cdot \mu_p(y(SL_2(\mathbb{Z}_p, p))) \cdot \mu_p(z(SL_2(\mathbb{Z}_p, p)))$$

$x(SL_2(\mathbb{Z}_p, p)) = 1 + \sum_{i=1}^{\infty} a_i p^i$, since μ_p is transitive invariant, $\mu_p(1 + \sum_{i=1}^{\infty} a_i p^i) = \mu_p(\sum_{i=1}^{\infty} a_i p^i) = \mu_p(p\mathbb{Z}_p)$, similarly, $\mu_p(y(SL_2(\mathbb{Z}_p, p))) = \mu_p(z(SL_2(\mathbb{Z}_p, p))) = \mu_p(p\mathbb{Z}_p)$, therefore

$$\omega_p(SL_2(\mathbb{Z}_p, p)) = 1 \cdot p^{-1} \cdot p^{-1} \cdot p^{-1}$$

So

$$\omega_p(SL_2(\mathbb{Z}_p)) = p(p^2 - 1)p^{-3} = 1 - p^{-2}$$

Remark 7. For general SL_n, we have the following partial calculation:

The dimension of SL_n is n^2-1, therefore $\omega_p(SL_n(\mathbb{Z}_p,p)) = p^{-n^2+1}$, and $|SL_n(\mathbb{F}_p)| = \frac{1}{p-1}\prod_{i=0}^{n-1}(p^n-p^i)$, so

$$\begin{aligned}
\omega_p(SL_2(\mathbb{Z}_p)) &= \frac{p^{-n^2+1}}{p-1}(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1}) && (1)\\
&= \frac{p^{-n^2+1}}{p-1}(p^n-1)p(p^{n-1}-1)p^2(p^{n-2}-1)\cdots p^{n-1}(p-1) && (2)\\
&= p^{-(n-1)(n+1)+(n-1)}(p^n-1)p(p^{n-1}-1)p^2(p^{n-2}-1)\cdots p^{n-2}(p^2-1) && (3)\\
&= p^{-n(n-1)}p^{\frac{(n-1)(n-2)}{2}}(p^n-1)(p^{n-1}-1)\cdots(p^2-1) && (4)\\
&= p^{-\frac{(n-1)(n+2)}{2}}(p^n-1)(p^{n-1}-1)\cdots(p^2-1) && (5)\\
&= p^{-n}(p^n-1)p^{-(n-1)}(p^{n-1}-1)\cdots p^{-2}(p^2-1) && (6)
\end{aligned}$$

Therefore

$$\prod_p \omega_p(SL_n(\mathbb{Z}_p)) = \zeta(2)^{-1}\zeta(3)^{-1}\zeta(4)^{-1}\cdots\zeta(n)^{-1} \tag{7}$$

3 Computing the Tamagawa Number of $\mathbf{SL}_2$

3.1 Reduction Theory for G_A relative to G_K

Definition 8. Reduction Theory Finding fundamental domain with global finiteness for lattices in Lie groups is called *reduction theory.*

Definition 9. Locally and Globally Finite A fundamental domain Ω for Γ in X is called *locally finite* if for every $x \in X$, there exists a neighborhood U such that U only meets finitely many Γ-translations of $\overline{\Omega}$. It is called *globally finite* if $\{\gamma \in \Gamma | \gamma\overline{\Omega} \cap \overline{\Omega} \neq \emptyset\}$ is finite.

Definition 10. [1, P163] A subset $F \subset G$ is a *fundamental domain* with respect to H if the restriction to F of the natural map $\pi : G \to G/H$ is bijective. This is equivalent to the following conditions:

1) $G = FH$

2) $F \cap Fh = \emptyset$ for any $h \neq e$ in H

2') $F \cap Fh$ has measure 0, for any $h \neq e$ in H

Definition 11. [1, P253] We call a subset Ω of G_A a *fundamental set* for G_K if

$(F1)_A$ $\Omega G_K = G_A$

$(F2)_A$ $\Omega^{-1}\Omega \cap G_K$ is finite

Theorem 1. *[1, P253] Let $G = GL_n$ over $\mathbb{Q}$ and let Σ be the fundamental set for $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$. Then $\Omega = \Sigma \times \prod_p G_{\mathbb{Z}_p}$ is a fundamental set for $G_{\mathbb{Q}}$ in G_A.*

Partial proof. Let's omit the proof of the fact that the class number of GL_n, $cl(G)$ is 1. Recall the definition of class number is

Definition 12. [1, P251] We call the h in the following decomposition of G_A into double cosets the *class number* of G and denote it by $cl(G)$.

$$G_A = \bigcup_{i=1}^{h} G_{A(\infty)} x_i G_K \tag{8}$$

We want to show $\Omega G_{\mathbb{Q}} = G_A$. We know $G_A = G_{A(\infty)} A_{\mathbb{Q}}$ since $cl(G) = 1$. Consider $\Omega G_{\mathbb{Z}} = (\Sigma \times \prod_p G_{\mathbb{Z}_p}) G_{\mathbb{Z}} = G_{\mathbb{R}} \times \prod_p G_{\mathbb{Z}_p} = G_{A(\infty)}$, therefore $\Omega G_{\mathbb{Q}} = \Omega G_{\mathbb{Z}} G_{\mathbb{Q}} = G_{A(\infty)} G_{\mathbb{Q}} = G_A$.

Next we want to show that $\Omega^{-1}\Omega \cap G_{\mathbb{Q}}$ if finite. Let $g \in \Omega^{-1}\Omega \cap G_{\mathbb{Q}}$, $g \in (\Sigma^{-1} \times \prod_p G_{\mathbb{Z}_p}^{-1} \cdot \Sigma \times \prod_p G_{\mathbb{Z}_p}) \cap G_{\mathbb{Q}}$, meaning g embeds diagonally in this intersection, thus $g \in G_{\mathbb{Z}_p}$. That means $g \in G_{\mathbb{Z}}$. The projection to the real component gives $g \in \Sigma^{-1}\Sigma$, since Σ is a fundamental set, we have the finiteness as a consequence. □

Remark 8. We want to make the same proof work the fundamental domain. One important thing to note is that with the definition of fundamental set, we can have the center of the group added onto Ω and it will still be a fundamental set. But if the center has non-zero measure, then it won't be a fundamental domain. Let's recall the definition for fundamental domain.

Claim 1. *Let $G = SL_2$ over $\mathbb{Q}$ and let Σ be a fundamental domain for $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$. Then $\Omega = \Sigma \times \prod_p G_{\mathbb{Z}_p}$ is a fundamental domain for $G_{\mathbb{Q}}$ in G_A.*

Proof. We use the fact that $cl(SL_2) = 1$. may need more care. One thought (perhaps wrong) is that $\mu(\Omega \cap \Omega h) = \omega_\infty(\Sigma \cap \Sigma h) \cdot \prod_p \omega_p(G_{\mathbb{Z}_p} \cap G_{\mathbb{Z}_p} h)$, since Σ is a fundamental domain, and the $\prod_p$ converges absolutely, the right hand side is 0. □

3.2 The Computation

Using all the tools we have obtained, we can go ahead and compute the Tamagawa number of SL_2.

By the Remark 2, we can concentrate on $\mathbb{Q}$ instead of arbitrary number field K. So let $G = SL_2$ over $\mathbb{Q}$. We would like to compute the invariant volume of $SL_2(A)/SL_2(\mathbb{Q})$, it is fortunate for us that we can do this through finding and integrating over the fundamental domain in $SL_2(A)$ of $SL_2(\mathbb{Q})$. Let F be the fundamental domain in $SL_2(\mathbb{R})$ of $SL_2(\mathbb{Z})$, by Claim 1 we have

$$\tau(SL_2) = \omega_\infty(F) \times \prod_p \omega_p(SL_2(\mathbb{Z}_p)) \tag{9}$$

By Example 3, we know

$$\prod_p \omega_p(SL(\mathbb{Z}_p)) = \prod_p (1 - p^{-2}) = \zeta(2)^{-1} \tag{10}$$

It remains to calculate F and $\omega_\infty(F)$. Consider the Iwasawa decomposition mentioned in Example 2 and the differential form determined by these coordinates. Therefore the volume of $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is expressed by

$$\int_F \alpha \; d\varphi \; d\alpha \; du \tag{11}$$

We shall construct F. To do so, we can think of F as the fundamental domain of $SL_2(\mathbb{Z})$ acting as a discrete transformation group on $SL_2(\mathbb{R})$. That leads us naturally to thinking about the fundamental domain of $SL_2(\mathbb{Z})$ acting on the upper half plane $\mathfrak{H} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$. That is the well-known fundamental domain

$$D = \{z \in SL_2(\mathbb{R})/SO_2(\mathbb{R}) : |\mathbb{R}e(z)| \leq \frac{1}{2}, |z| \geq 1\} \tag{12}$$

Let the projection map $\varphi : SL_2(\mathbb{R}) \to \mathfrak{H}$ be

$$\varphi : \begin{pmatrix} x & y \\ u & t \end{pmatrix} \mapsto \frac{ti + y}{ui + x} \tag{13}$$

Then it is easy to see that for F we may take $F = K_0 D_0$, where

$$K_0 = \{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} : \varphi \in [0, \pi]\} \tag{14}$$

$$D_0 = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : a > 0, \varphi \begin{pmatrix} a & au \\ 0 & a^{-1} \end{pmatrix} \in D\} \tag{15}$$

A simple calculation shows that

$$a > 0, \varphi \begin{pmatrix} a & au \\ 0 & a^{-1} \end{pmatrix} \in D \Leftrightarrow |u| \leq \frac{1}{2}, 0 \leq a \leq \frac{1}{\sqrt[2]{1-u^2}} \tag{16}$$

Therefore the volume of $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is

$$\omega_\infty(SL_2(\mathbb{R})/SL_2(\mathbb{Z})) = \int_0^\pi d\varphi \int_{-1/2}^{1/2} du \int_0^{1/\sqrt[4]{1-u^2}} \alpha \; d\alpha = \frac{\pi^2}{6} = \zeta(2) \tag{17}$$

By equation 9 we have

$$\tau(SL_2) = \zeta(2) \times \zeta(2)^{-1} = 1 \tag{18}$$

4 Some General Results

4.1 Tamagawa Number of SL_n

Theorem 2. *[2, Theorem 14.4]*

$$\omega_\infty(SL_n(\mathbb{R})/SL_n(\mathbb{Z})) = \zeta(2)\zeta(3)\cdots\zeta(n) \tag{19}$$

Combining this theorem with Example 7, we can see that $\tau(SL_n) = 1$.

4.2 Tamagawa Number for Reductive Groups

[1, P263] The definition of Tamagawa numbers requires some modification for reductive groups, since for many cases important in applications (such as the 1-dimensional split torus $\mathbb{G}_m$) the volume G_A/G_K is infinite. This leads us to think of other homogeneous spaces that are closely related to adele groups but have finite invariant volume. Since the obstruction to the finiteness of the volume of G_A/G_K comes from the existence of non-trivial K-characters [1, Theorem 5.5], we associate each character $\chi \in X(G)_K$ the continuous homomorphism $c_K(\chi) : G_A \to \mathbb{R}^{>0}$ given by

$$c_K(\chi)((g_\nu)) = \prod_\nu |\chi(g_\nu)|_\nu \tag{20}$$

Then we define

$$G_A^{(1)} = \bigcap_{\chi \in X(G)_K} \ker c_K(\chi) \tag{21}$$

We have the following theorem

Theorem 3. *Let G be a connected K-group. Then $G_A^{(1)}$ is unimodular and $G_A^{(1)}/G_K$ has finite invariant volume.*

4.3 Weil Conjecture

[1, P263]

Theorem 4. *If G is semi-simple simply connected, then $\tau(G) = 1$.*

Weil developed a method of computing Tamagawa numbers using induction, the residues of some analogs of the zeta function, and the Poisson summation formula. This method allows one to prove Weil conjecture for many classical groups and some exceptional groups. Later Mars computed the Tamagawa number for unitary groups of type A_n and thereby completed the proof of the Weil conjecture for classical semi-simple groups over number fields. A unified proof of the conjecture for Chevalley groups was given by Langlands 1966. Lai 1976, 1980 computed $\tau(G)$ for G quasi-split. A complete proof of the conjecture was obtained by Kottwitz 1988 modulo the validity of the Hasse principle for Galois cohomology of simply connected semisimple algebraic groups. Chernousov 1989, completed the proof of this Hasse principle for groups of type E_8, Thus Weil conjecture has been proved.

Why do we care about simply connected group? See the following elegant result:

Let G be a semi-simple K-group, let $\pi : \widetilde{G} \to G$ be the universal K-covering, let $F = \ker \pi$ be the fundamental group of G, and $X(F)$ be its group of characters. Then

$$\tau(G) = \tau(\widetilde{(G)}) \frac{h^0(X(F))}{i^1(X(F))} \tag{22}$$

where

$$h^0(X(F)) = [H^0(K, X(F))] = [X(F)_K] \tag{23}$$

and $i^1(X(F))$ is the order of the kernel of the canonical map $H^1(K, X(F)) \to \prod_\nu H^1(K_\nu, X(F))$. Thus it is suffice to compute $\tau(G)$ for G simply connected.

5 Some Definitions and Well Known Facts

Definition 13 (Module of G)**.** [1, P159] For $x \in G$, let $\Delta_G(x)$ denote the module of the corresponding inner automorphism $Inn(x) : g \mapsto xgx^{-1}$. The function $\Delta_G : G \to \mathbb{R}^+$ is called the *module* of G and is a continuous homomorphism.

Definition 14 (Unimodular)**.** [1, P160] If $\Delta_G \equiv 1$ then G is said to be *unimodular*.

Remark 9. It is the uniqueness of Haar measure that enable us to define the module. On the other hand, the unimodularity is rather a property of G than of the Haar measure μ.

Definition 15. Let X be a topological space, X is called locally compact if every point of X has a compact neighborhood.

Example 4. The real field $\mathbb{R}$ is locally compact. p-adic numbers $\mathbb{Q}_p$ is locally compact because it is homeomorphic to the Cantor set (which is compact) minus one point. The rational numbers $\mathbb{Q}$ (provided to topology from $\mathbb{R}$) is not locally compact.

Remark 10. Locally compact groups are important because they have a natural measure called the Haar measure.

Theorem 5. *[1, P159] Let G be a locally compact group. Then there is a left (right) Haar measure on G, which is unique up to multiplication by a positive constant.*

Remark 11. Note that if μ is a left Haar measure on G, then $\hat{\mu}$, given by $\hat{\mu}(X) = \mu(X^{-1})$ for all $X \subset G$ such that X^{-1} is μ-measurable, is a right Haar measure on G. Therefore the assertion above for left and right Haar measures are equivalent.

5.1 Adeles, approximation, local-global principle

Proposition 1 (Product Formula)**.** [1, P12] For any $a \in K^*$, we have $\prod_{\nu \in V^K} |a|_\nu^{n_\nu} = 1$, where $n_\nu = [K_\nu : \mathbb{Q}_p]$ (same definition if change $\mathbb{Q}_p$ into $\mathbb{R}$) is the local dimension with respect to the p-adic valuation $| \ |_\nu$. Normalization gives $||a||_\nu = |a|_\nu^{n_\nu}$ and the product formula can be stated more elegantly as $\prod_{\nu \in V^K} ||a||_\nu = 1$.

6 Notations

- K will always be an algebraic number field, i.e., a finite extension of $\mathbb{Q}$
- K_ν is the completion of K with respect to a valuation $| \ |_\nu$
- $\mathcal{O}_\nu$ is the ring of integers in K_ν.
- V^K is the equivalent classes of valuations on K
- V_f^K is the set of non-archimedean valuations obtained as extensions of the p-adic valuation $| \ |_p$ of $\mathbb{Q}$, for each prime number p. V_∞^K is the archimedean valuation, $| \ |_\infty$ is the ordinary absolute value on $\mathbb{Q}$.

  Remark 12. $V^K = V_f^K \cup V_\infty^K \cup | \ |_\infty$
- ω denotes a rational differential K-form
- ω_ν is the left-invariant measure induced by ω on G_{K_ν} for every $\nu \in V^K$

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